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KERR GEOMETRY XI.

KERR-NEWMAN INTERIOR

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Contents

Recently we found an interior for the Kerr metric, closely related to the Schwarzschild interior. An extension to the Kerr-Newman metric is straight forward.

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1. INTRODUCTION

In a previous paper [1] we have re-investigated the Kerr-Newman model. We have worked out a covariant decomposition of the field equations, and we have calculated the stress-energy tensor for some preferred reference systems. In this paper we match an interior to the KN metric. As the methods are the same we have used for our new Kerr interior [2,3] we only present the most important results.

In Sec. 2 we investigate the nonrotating interior seed metric and we define the basic quantities. We refer to the possibility that the seed metric can be interpreted as metric of a surface endowed with nonholonomicity and embedded in a 5-dimensional flat space. In Sec. 3 we note the field strengths of the seed metric and in Sec. 4 we implement the rotation.

2. METRIC AND SURFACE

To consider 4-dimensional surfaces embedded in a higher dimensional flat space is a powerful tool for finding new solutions of the Einstein field equations. In this way, we have found a new interior for the Kerr field, in another paper [1] we have prepared the KN exterior by embedding methods for supporting it by an interior.

We start with the nonrotating seed metric of a surface. That is the raw construction for the KN interior

$$ds^2 = \alpha_I^2 a_R^2 dr^2 + \Lambda^2 d\vartheta^2 + \sigma^2 d\varphi^2 - a_T^2 dt^2 \ . \qquad (1.1)$$

The metric is based on the elliptic-hyperbolic Boyer-Lindquist co-ordinate system. The anholonomic differentials read as

$$dx^1 = \alpha_I a_R dr, \quad dx^2 = \Lambda d\vartheta, \quad dx^3 = \sigma d\varphi, \quad dx^4 = a_T i dt \ . \qquad (1.2)$$

We interpret the space-like part of the metric as the metric of a surface with elliptical parallels and circular parallels with radii

$$\sigma = A \sin\vartheta \qquad (1.3)$$

The minor-semi axes of the ellipses are labeled by r, the major-semi axes by A with

$$A^2 = r^2 + a^2 \ . \qquad (1.4)$$

The eccentricity of the confocal ellipses are a. The *elliptical factor* is

$$a_R = \frac{\Lambda}{A}, \quad \Lambda^2 = r^2 + a^2 \cos^2 \vartheta \ , \tag{1.5}$$

while the *gravitational factors* are

$$\begin{gathered} \alpha_I = \frac{1}{a_I}, \quad a_I^2 = 1 - \frac{r^2}{\mathcal{R}^2}, \quad a_T = \frac{1}{2}\left[\left(1 + 2\Phi_g^2\right)\cos\eta_g - \cos\eta\right]\Phi_g^{-2} \\ 2\Phi_g^2 = \frac{A_g^2}{r_g}\frac{2Mr_g - e^2}{M\left(r_g^2 - a^2\right) - e^2 r_g}, \quad \cos\eta_g = \sqrt{1 - \frac{r_g^2}{\mathcal{R}_g^2}}, \quad \cos\eta = \sqrt{1 - \frac{r^2}{\mathcal{R}^2}} = a_I \end{gathered} \ . \tag{1.6}$$

$\mathcal{R}$ is a variable of the local extra dimension and parameterizes a family of surfaces with elliptical shape. These factors are responsible for the curvature of the interior KN surface. The subscript g denotes the constant values of the variables on the boundary surface connecting the interior and exterior solution. Selecting one of these surfaces by the embedding condition $\mathcal{R} = \mathcal{R}_g = \text{const.}$ and the junction condition

$$\mathcal{R}_g = \frac{A_g}{\sqrt{\frac{2M}{r_g} - \frac{e^2}{r_g^2}}} \tag{1.7}$$

we can match the interior solution to the exterior if we also regard

$$\eta_g = -\varepsilon_g \tag{1.8}$$

on the boundary. Since the interior and exterior solutions use their own co-ordinate system, the two angles have opposite signs. η_g is the aperture angle at the minor-semi axis of the interior surface and $\eta < \eta_g$ is the quasi-polar angle to a point of this surface. In paper [1] we have found the curvature radius of the exterior surface to be

$$\rho_S = a_R A^3 \frac{\sqrt{2Mr - e^2}}{M\left(r^2 - a^2\right) - e^2 r} \ . \tag{1.9}$$

Its value on the boundary surface of the geometries and at the semi-minor axis is

$$\rho_g = A_g^3 \frac{\sqrt{2Mr_g - e^2}}{M\left(r_g^2 - a^2\right) - e^2 r_g} \ . \tag{1.10}$$

Substituting (1.7) into this relation we obtain

$$\rho_g = 2\mathbb{R}_g\Phi_g^2 \ . \qquad (1.11)$$

If only the Kerr parameter a is put zero one obtains the Reissner-Nordström interior recently found by us in [4]. For the Schwarzschild case a = 0, e = 0 the auxiliary quantity reduces to $\Phi_g = 1$ and $\rho_g = \sqrt{\frac{2r^3}{M}}$ is the curvature radius of the Schwarzschild parabola. The gravitational factor $a_T = \frac{1}{2}\left[3\cos\eta_g - \cos\eta\right]$ is the well-known time factor of the Schwarzschild interior. The space-like part of the Schwarzschild interior is the cap of a sphere with radius $\mathbb{R}$. For the Kerr and Kerr-Newman cases the surface behaves Schwarzschild-like only at the minor-semi axes of the elliptical slices of the surface. $\mathbb{R}$ is the radius of a circle in this place. Otherwise, one has to multiply $\mathbb{R}$ with the elliptical factor to get the curvature radii $\mathbb{R}a_R$ of the surface.

To understand the time-like part of the metric, we have to use the substitution

$$idt = \rho_g di\psi \ , \qquad (1.12)$$

where ρ_g is the exterior curvature radius of the radial lines on the boundary surface of the two geometries.

From (1.6), (1.11) and (1.12) we get

$$dx^4 = \left[\left(\mathbb{R}_g + \rho_g\right)\cos\eta_g - \mathbb{R}_g\cos\eta\right]di\psi \ . \qquad (1.13)$$

The flow of time is proportional to the area of a ring sector of two pseudo circles (hyperbolae of constant curvature). We have presented this problem in detail in paper [2,3]. There we have used the theory of double surfaces to embed the Kerr interior into a 5-dimensional flat space.

3. THE FIELD STRENGTHS OF THE SEED METRIC

In the following, we discuss some geometrical properties of the seed metric. From (1.2) we read the 4-bein field and we calculate the Ricci-rotation coefficients, and we also separate the expressions for the different curvatures of the surface

$$A_{mn}{}^s = B_{mn}{}^s + N_{mn}{}^s + C_{mn}{}^s + E_{mn}{}^s \qquad (2.1)$$

With the unit vectors m_n= {1,0,0,0} , b_n= {0,1,0,0}, c_n= {0,0,1,0} , u_n= {0,0,0,1} we are able to perform the decompositions

$$
\begin{aligned}
&B_{mn}{}^{s} = b_m B_n b^s - b_m b_n B^s, \quad B_n = \left\{ \frac{a_I}{\rho_E}, 0,0,0 \right\} \\
&N_{mn}{}^{s} = m_m N_n m^s - m_m m_n N^s, \quad N_n = \left\{ 0, \frac{1}{\rho_H}, 0,0 \right\} \\
&C_{mn}{}^{s} = c_m C_n c^s - c_m c_n C^s, \quad C_n = \left\{ a_I \frac{r}{A\Lambda}, \frac{1}{\Lambda} \cot\vartheta, 0,0 \right\} \\
&E_{mn}{}^{s} = -\left[u_m E_n u^s - u_m u_n E^s \right], \quad E_n = \left\{ -\frac{1}{\rho_g^S a_T} \sin\eta, 0,0,0 \right\}, \ \sin\eta = \frac{r}{\mathcal{R}}
\end{aligned}
\quad . \tag{2.2}
$$

The curvature vectors[1] of the Boyer-Lindquist ellipses and their hyperbolic orthonormal trajectories are

$$
\rho_E = \frac{\Lambda^3}{rA}, \ \ \rho_H = -\frac{\Lambda^3}{a^2 \sin\vartheta \cos\vartheta} \ . \tag{2.3}
$$

The curvature vectors of the radial integral lines of the surface are

$$
\rho_g^S(r,\vartheta) = \rho_g a_R(r,\vartheta) \ . \tag{2.4}
$$

On the boundary surface one has

$$
a_T^g = \cos\eta_g = \cos\varepsilon_g, \quad \sin\eta_g = -\sin\varepsilon_g \ . \tag{2.5}
$$

Thus, the gravitational force coincides with the exterior force on the boundary.

$$
E_1^g = -\frac{1}{\rho_g^S} \tan\eta_g = \frac{1}{\rho_g^S} \tan\varepsilon_g
$$

And so do the other curvature quantities in (2.2). By use of (1.7) one can show that

$$
a_I^g = \frac{\delta_g}{A_g} = a_S^g, \quad \delta_g^2 = r_g^2 - 2Mr_g + a^2 + e^2 \tag{2.6}
$$

takes the same value as the quantity defined in [1] for the exterior solution. As we have explained in previous papers, the field equations of the exterior and interior geometry of the Kerr family can be described with the help of five dimensions. The 5-dimensional theory provides an economical representation for the field equations, but we are not going to repeat all that. We restrict ourselves to supplementing the curvature equations with

[1] To be more precise, the only remaining components of the curvature vector in the reference system in use.

$$B_0 = \frac{v_I}{\rho_E}, \quad C_0 = v_I \frac{r}{A\Lambda}, \quad E_0 = \frac{1}{\rho_g^S a_T} \cos\eta, \quad v_I = \sin\eta \tag{2.7}$$

and we add a new quantity

$$M_0 = \frac{1}{a_R \mathcal{R}} \tag{2.8}$$

describing the extrinsic curvature of the surface. One can compose the field equations of the seed metric

$$\begin{aligned}
&-\left[B_{n\underset{2}{||}m} - B_{n\underset{2}{||}s} b^s b_m + B_n B_m\right] - b_n b_m \left[B^s{}_{\underset{2}{||}s} + B^s B_s\right] \\
&-\left[N_{n\underset{2}{||}m} - N_{n\underset{2}{||}s} m^s m_m + N_n N_m\right] - m_n m_m \left[N^s{}_{\underset{2}{||}s} + N^s N_s\right] \\
&-\left[C_{n\underset{3}{||}m} + C_n C_m\right] - c_n c_m \left[C^s{}_{\underset{3}{||}s} + C^s C_s\right] \\
&+\left[E_{n\underset{4}{||}m} - E_n E_m\right] + u_n u_m \left[E^s{}_{\underset{4}{||}s} - E^s E_s\right] = -\kappa\left(T_{mn} - \frac{1}{2} g_{mn} T\right)
\end{aligned} \quad . \tag{2.9}$$

with the help these extra components. The graded covariant derivatives therein we have defined in former papers and they have the properties

$$\begin{aligned}
m_{m\underset{1}{||}n} = m_{m|n} = 0, \quad b_{m\underset{2}{||}n} = b_{m|n} = 0, \quad c_{m\underset{3}{||}n} = c_{m|n} - B_{nm}{}^s c_s - N_{nm}{}^s c_s = 0 \\
u_{m\underset{4}{||}n} = u_{m|n} - B_{nm}{}^s u_s - N_{nm}{}^s u_s - C_{nm}{}^s u_s = 0
\end{aligned} \quad . \tag{2.10}$$

The right side of the field equations takes the form

$$\begin{aligned}
\kappa\left(T_{mn} - \frac{1}{2} g_{mn} T\right) = &-m_m m_n \left(M_0 B_0 + M_0 C_0 - M_0 E_0\right) \\
&- b_m b_n \left(B_0 M_0 + B_0 C_0 - B_0 E_0\right) \\
&- c_m c_n \left(C_0 M_0 + C_0 B_0 - C_0 E_0\right) \\
&+ u_m u_n \left(E_0 M_0 + E_0 B_0 + E_0 C_0\right) \\
&- \left(m_m m_n + b_m b_n\right) \tilde{\Omega}^{s3} \tilde{\Omega}_{3s} \sin^2\eta \\
&+ \left(m_m m_n + c_m c_n\right) N_s C^s \sin^2\eta + 2 N_{(m} E_{n)} \\
&+ \left(m_m m_n + u_m u_n\right) E_s F^s
\end{aligned} \quad . \tag{2.11}$$

The first four lines of these equations contain the generalized second fundamental forms of the theory of surfaces. The other terms have their origin in the elliptical structure of this model

$$\tilde{\Omega}^{s3}\tilde{\Omega}_{3s} = \tilde{H}_{13}\tilde{H}_{13} - \tilde{H}_{23}\tilde{H}_{23}, \quad \tilde{H}_{s3} = i\alpha_R^2\omega\left\{\frac{r}{\Lambda}\sin\vartheta, \frac{A}{\Lambda}\cos\vartheta, 0, 0\right\}, \quad \omega = \frac{a}{A^2} \ , \tag{2.12}$$

while

$$F_s = \alpha_R^2\omega^2\sigma\sigma_{|s} \tag{2.13}$$

is the centrifugal force as we will see in the next Section. Evidently, the seed metric does not provide vacuum field equations. Although the seed metric is an exact solution of the Einstein field equations the model has no physical meaning.

4. THE ROTATING MODEL

From the seed metric we derive the rotating metric by an intrinsic transformation. The surface described in the last section does not change by this transformation anyway, but an additional structure on the surface is invoked by rotating some vectors of the local 4-bein. With the anholonomic transformation

$$\begin{aligned}
&\Lambda_3^{3'} = \alpha_R, \quad \Lambda_4^{3'} = i\alpha_R\omega, \quad \Lambda_3^{4'} = -i\alpha_R\omega\sigma^2, \quad \Lambda_4^{4'} = \alpha_R \\
&\Lambda_{3'}^{3} = \alpha_R, \quad \Lambda_{4'}^{3} = -i\alpha_R\omega, \quad \Lambda_{3'}^{4} = i\alpha_R\omega\sigma^2, \quad \Lambda_{4'}^{4} = \alpha_R
\end{aligned} \tag{4.1}$$

one obtains the new components of the 4-bein field with

$$\overset{m}{e}_i = \Lambda_i^{i'}\overset{m}{e}_{i'}, \quad \underset{m}{e}^i = \Lambda_{i'}^{i}\underset{m}{e}^{i'}, \quad g_{ik} = \Lambda_{ik}^{i'k'}g_{i'k'} \ . \tag{4.2}$$

The new line element reads as

$$ds^2 = dx^{1^2} + dx^{2^2} + \left[\alpha_R dx^3 + i\alpha_R\omega\sigma dx^4\right]^2 + a_T^2\left[-i\alpha_R\omega\sigma dx^3 + \alpha_R dx^4\right]^2, \quad dx^1 = \alpha_I a_R dr \ . \tag{4.3}$$

As a consequence of the transformation (4.1) new quantities appear in the Ricci-rotation coefficients. These are the Coriolis-like field strengths H_{mn}^T and the shears D_{mn}^T

$$\begin{aligned}
&H_{mns} = -\Omega_{nm}^T u_S + \Omega_{sm}^T u_n + \Omega_{sn}^T u_m + \alpha_T D_{ns} u_m \\
&\Omega_{nm}^T = -\left[H_{mn}^T + D_{mn}^T\right], \quad H_{mn}^T = a_T\left[H_{mn} + D_{[mn]}\right], \quad D_{mn}^T = \alpha_T D_{(mn)} \ . \\
&H_{m3} = i\alpha_R^2\omega\sigma_{|m}, \quad D_{m3} = i\alpha_R^2\omega_{|m}\sigma, \quad D_{3m} = 0
\end{aligned} \tag{4.4}$$

Moreover, one has

$$C_m \to C^T_m = C_m + F_m, \quad E_m \to E^T_m = E_m + F_m \ . \tag{4.5}$$

F is the centrifugal force defined in (2.13). With these new quantities one obtains field equations formally equivalent to the field equations derived for the Kerr interior [2,3]. The stress-energy tensor is

$$\begin{aligned}
\kappa T_{11} &= -M_0 B_0 + 2B_0 C_0 - 2B_0 E_0 - \left[\Omega^{\alpha 3}_{C}\Omega^{C}_{3\alpha} - \Omega^{\alpha 3}_{T}\Omega^{T}_{3\alpha}\right] \\
\kappa T_{22} &= -M_0 E_0 + M_0 B_0 + 2M_0 C_0 - 2B_0 C_0 - C_0 E_0 + 2F_0 E_0 + \left[\Omega^{\alpha 3}_{C}\Omega^{C}_{3\alpha} - \Omega^{\alpha 3}_{T}\Omega^{T}_{3\alpha}\right] \\
\kappa T_{33} &= -M_0 E_0 + M_0 B_0 + 2M_0 C_0 - 2B_0 C_0 - C_0 E_0 + 2F_0 E_0 \\
&\quad + \left[\Omega^{\alpha 3}_{C}\Omega^{C}_{\alpha 3} - \Omega^{\alpha 3}_{T}\Omega^{T}_{\alpha 3}\right] + \left[\Omega^{\alpha\beta}_{C}\Omega^{C}_{\beta\alpha} - \Omega^{\alpha\beta}_{T}\Omega^{T}_{\beta\alpha}\right] \\
\kappa T_{44} &= 2B_0 C_0 + M_0 B_0 + \left[\Omega^{\alpha 3}_{C}\Omega^{C}_{\alpha 3} - \Omega^{\alpha 3}_{T}\Omega^{T}_{\alpha 3}\right] - \left[\Omega^{\alpha\beta}_{C}\Omega^{C}_{\beta\alpha} - \Omega^{\alpha\beta}_{T}\Omega^{T}_{\beta\alpha}\right] \\
\kappa T_{34} &= \Omega^{T}_{03} M_0 + \Omega^{T}_{03} E_0 - \left[1 - \frac{a_T^2}{a_I^2}\right]\Omega^{T}_{13} B_1 - \left[1 - \frac{a_I^2}{a_T^2}\right]\Omega^{T}_{23} N_2
\end{aligned} \ . \tag{4.6}$$

The auxiliary quantities

$$F_0 = \alpha_R^2 \omega^2 \sigma \sigma_{|0}, \quad \Omega^{T}_{03} = i\alpha_T \alpha_R^2 \omega \sigma_{|0}, \quad \sigma_{|0} = \frac{r}{\Lambda}\sin\vartheta \sin\eta \tag{4.7}$$

are the components with respect to the extra dimension of the corresponding 4-dimensional quantities. $\Omega^{C}_{\alpha\beta}$ are the rotational field strengths

$$\Omega^{C}_{nm} = -\left[H^{C}_{mn} + D^{C}_{mn}\right], \quad H^{C}_{mn} = a_I\left[H_{mn} + D_{[mn]}\right], \quad D^{C}_{mn} = \alpha_I D_{(mn)} \ , \tag{4.8}$$

discussed in [1, 2, 3]. If we insert the boundary values (2.5) and (2.6) into (4.6) the brackets in (4.6) vanish and the stress-energy tensor can be split into

$$T^{g}_{mn} = T^{e}_{mn} + T^{r}_{mn} \ , \tag{4.9}$$

where

$$\kappa T^{e}_{mn} = \begin{pmatrix} \mathcal{F}^2 & & & \\ & -\mathcal{F}^2 & & \\ & & -\mathcal{F}^2 & \\ & & & \mathcal{F}^2 \end{pmatrix}, \quad \mathcal{F} = \mathcal{F}_{14} = \frac{e}{\Lambda^2} \tag{4.10}$$

coincides with the electric stress-energy tensor of the exterior solution. There is no hydrostatic pressure[2] $T^{r}_{11} = 0$, but there are stresses $T^{r}_{22} = T^{r}_{33} \neq 0$. The jumps in

$$\kappa T^{r}_{34} = \Omega^{Tg}_{03}\left(M^{g}_{0} + E^{g}_{0}\right), \quad \kappa T^{r}_{44} = B^{g}_{0}\left(M^{g}_{0} + E^{g}_{0}\right)$$

are easily understood. M^{g}_{0} and E^{g}_{0} are the radial curvatures of the interior and exterior solutions, both being present on the boundary surface. Since $M^{g}_{0} + E^{g}_{0} = \frac{1}{a^{g}_{R}}\left(\frac{1}{\mathcal{R}} + \frac{1}{\rho_g}\right)$ is positive, the energy is also positive. For the exterior solution the corresponding terms vanish because $M^{ext}_{0} = -E^{ext}_{0}$.

5. CONCLUSIONS

We have shown that the construction of an interior solution matching the Kerr-Newman exterior solution is possible. We have used geometrical methods based on embedding the metric into a 5-dimensional flat space, a method which had been successfully used for other members of the Kerr family. Putting zero the parameters e or a or both one obtains a Kerr interior, an interior for the Reissner-Nordström solution or the Schwarzschild interior solution. Although there won't be any stellar objects having the properties of this solution the interior metric is of some interest as it supplements other solutions of the Kerr family.

6. REFERENCES

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[2] $M_0B_0 - 2B_0C_0 + 2B_0E_0 = -\mathcal{F}^2$ on the boundary surface.