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KERR GEOMETRY IX. THE GEOMETRICAL IMPLEMENTATION OF ROTATION

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The problem of rotation is re-examined for the exterior and interior Kerr solution. Starting with the static seed metric we subject the 4-bein to an intrinsic transformation, and we obtain both Kerr solutions.

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1. INTRODUCTION

For some decades many searchers [1-10] have tried to explain the Kerr metric by means of geometrical or physical methods. We make a new attempt by using basic methods. We decompose the metric into tetrads and we calculate the lengths of the tangent and gradient vectors and the angles enclosed by these vectors.

Thus, we can show that the Kerr interior and the Kerr exterior metric may be derived from a static seed metric by a transformation that rotates the third and fourth bein vector and in addition changes their lengths. The rotational part of the metric with its cross term appears as law of cosine.

In Sec. 2 we reformulate the Carter tetrad decomposition of the metric, and we transform the metric into a very compact form. In Sec. 3 we analyze the transformation to the representation of Iyer and Kumar, and Bardeen. In Sec. 4 we discuss some related problems.

2. ANALYZING THE KERR METRIC

In a former paper [11] we have proposed a rather compact form of the Kerr metric. We have used the Boyer-Lindquist co-ordinate system and the tetrad representation of Carter [3]:

$$ds^2 = dx^{1^2} + dx^{2^2} + \left[\alpha_R dx^3 + i\alpha_R \omega\sigma dx^4\right]^2 + a_S^2\left[-i\alpha_R\omega\sigma dx^3 + \alpha_R dx^4\right]^2 . \qquad (2.1)$$

We use the following abbreviations

$$dx^1 = \alpha_S a_R dr,\ dx^2 = \Lambda d\vartheta,\ dx^3 = \sigma d\varphi,\ dx^4 = idt\ , \qquad (2.2)$$

$$A^2 = r^2 + a^2,\quad \Lambda^2 = r^2 + a^2\cos^2\vartheta,\quad \alpha_R = \frac{A}{\Lambda},\quad a_R = \frac{\Lambda}{A},\quad \omega = \frac{a}{A^2},\quad \sigma = A\sin\vartheta\ , \qquad (2.3)$$

$$a_S = \frac{\delta}{A},\quad \alpha_S = \frac{A}{\delta},\quad \delta^2 = r^2 + a^2 - 2Mr,\quad \partial_1 = a_S\alpha_R\frac{\partial}{\partial r}\ . \qquad (2.4)$$

A and r are the major semi-axes and minor semi-axes of confocal ellipses with eccentricity a, ω the observer's angular velocity, σ the observer's distance from the rotation axis and α_R the Lorentz factor of this rotation. From (2.1) we read the components of the 4-bein fields.

$$\begin{array}{l} \overset{1}{e}_1 = \alpha_S a_R,\ \overset{2}{e}_2 = \Lambda,\ \overset{3}{e}_3 = \alpha_R\sigma,\ \overset{3}{e}_4 = i\alpha_R\omega\sigma,\ \overset{4}{e}_3 = -i a_S\alpha_R\omega\sigma^2,\ \overset{4}{e}_4 = a_S\alpha_R \\ \underset{1}{e}^1 = a_S\alpha_R,\ \underset{2}{e}^2 = \frac{1}{\Lambda},\ \underset{3}{e}^3 = \frac{\alpha_R}{\sigma},\ \underset{3}{e}^4 = i\alpha_R\omega\sigma,\ \underset{4}{e}^3 = -i\alpha_S\alpha_R\omega,\quad \underset{4}{e}^4 = \alpha_S\alpha_R \end{array} . \qquad (2.5)$$

The $\overset{m}{e}_i, \underset{m}{e}^i$ are the components of an anholonomic orthogonal reference system

$$g^{ik}\overset{m}{e}_i\overset{n}{e}_k = g^{mn} = \delta^{mn}, \quad g_{ik}\underset{m}{e}^i\underset{n}{e}^k = g_{mn} = \delta_{mn} . \qquad (2.6)$$

The indices m, n enumerate the vectors, the i, k their components with respect to an oblique angled co-ordinate system. The tangent vectors of this oblique-angled co-ordinate system are the $\underset{i}{h}^m$, and the gradient vectors the $\overset{i}{h}_m$, where the i's now are the labels of the vectors and the m's are the components of these vectors with respect to the local Cartesian co-ordinate system. The components of these vectors are *numerically* equivalent to the components of the above-defined vectors

$$\overset{i}{h}_m \stackrel{*}{=} \underset{m}{e}^i, \quad \underset{i}{h}^m \stackrel{*}{=} \overset{m}{e}_i . \qquad (2.7)$$

The oblique-angled vectors are not unit vectors. We calculate their lengths as

$$\begin{array}{ll} \tau_3 = \sqrt{\underset{3}{h}^m\underset{3}{h}^m} = a_{BC}\alpha_R\sigma, & \tau_4 = \sqrt{\underset{4}{h}^m\underset{4}{h}^m} = a_{AC}\alpha_R a_S \\ \gamma^3 = \sqrt{\overset{3}{h}_m\overset{3}{h}_m} = a_{AC}\alpha_R\frac{1}{\sigma}, & \gamma^4 = \sqrt{\overset{4}{h}_m\overset{4}{h}_m} = a_{BC}\alpha_R\alpha_S \end{array} , \qquad (2.8)$$

wherein

$$\begin{array}{lll} a_{AC}^2 = 1-\omega_{AC}^2\sigma^2, & \alpha_{AC} = 1/a_{AC}, & \omega_{AC} = \alpha_S\omega \\ a_{BC}^2 = 1-\omega_{BC}^2\sigma^2, & \alpha_{BC} = 1/a_{BC}, & \omega_{BC} = a_S\omega \end{array} . \qquad (2.9)$$

The angle enclosed by the tangent vectors is

$$\cos i\beta'_{AB} = \frac{\underset{3}{h}^3\underset{4}{h}^3 + \underset{3}{h}^4\underset{4}{h}^4}{\tau_3\tau_4} = i\alpha_{AC}\alpha_{BC}\left(\omega_{AC}-\omega_{BC}\right)\sigma = i\alpha_{AB}\omega_{AB}\sigma \qquad (2.10)$$

and

$$\alpha_{AB} = \frac{1}{\sqrt{1-\omega_{AB}^2\sigma^2}} = \alpha_{AC}\alpha_{BC}\left(1-\omega_{AC}\omega_{BC}\sigma^2\right) = \alpha_{AC}\alpha_{BC}a_R^2$$
$$\omega_{AB} = \frac{\omega_{AC}-\omega_{BC}}{1-\omega_{AC}\omega_{BC}\sigma^2} = \alpha_R^2\left(\omega_{AC}-\omega_{BC}\right) \quad . \qquad (2.11)$$

For the 'complementary' angle $i\beta_{AB}$ of $i\beta'_{AB}$ we get

$$\cos i\beta_{AB} = \alpha_{AB}, \quad \sin i\beta_{AB} = i\alpha_{AB}\omega_{AB}\sigma \ . \qquad (2.12)$$

Suppressing the first two dimensions we obtain for the Carter tetrads

$$\overset{m}{e}_3 \doteq \underset{3}{h}^m = \{\cos i\beta_{BC}, -\sin i\beta_{BC}\}\tau_3, \quad \overset{m}{e}_4 \doteq \underset{4}{h}^m = \{\sin i\beta_{AC}, \cos i\beta_{AC}\}\tau_4$$
$$\underset{m}{e}^3 \doteq \overset{3}{h}_m = \{\cos i\beta_{AC}, -\sin i\beta_{AC}\}\gamma^3, \quad \underset{m}{e}^4 \doteq \overset{4}{h}_m = \{\sin i\beta_{BC}, \cos i\beta_{BC}\}\gamma^4 \ , \qquad (2.13)$$
$$\cos i\beta_{AC} = \alpha_{AC}, \quad \sin i\beta_{AC} = i\alpha_{AC}\omega_{AC}\sigma$$
$$\cos i\beta_{BC} = \alpha_{BC}, \quad \sin i\beta_{BC} = i\alpha_{BC}\omega_{BC}\sigma$$

which demonstrates that the oblique-angled bein vectors are derived from an orthogonal unit vector system by rotating the tangent and gradient vectors through the angles $i\beta_{AC}$ and $i\beta_{BC}$ and stretching the vectors by the factors τ and γ. The τ and γ are the local measures of the oblique-angled tangent and gradient vectors. The components of the metric read as

$$g_{33} = \tau_3\tau_3, \quad g_{34} = i\alpha_{AB}\omega_{AB}\sigma\tau_3\tau_4, \quad g_{44} = \tau_4\tau_4$$
$$g^{33} = \gamma^3\gamma^3, \quad g^{34} = -i\alpha_{AB}\omega_{AB}\sigma\gamma^3\gamma^4, \quad g^{44} = \gamma^4\gamma^4 \ . \qquad (2.14)$$

At the end, we get for the rotational part of the Kerr line element the law of cosine

$$ds^2 = \left(\tau_3 dx^3\right)^2 - 2\cos i\beta''_{AB}\left(\tau_3 dx^3\right)\left(\tau_4 dx^4\right) + \left(\tau_4 dx^4\right)^2, \quad dx^3 = d\varphi \ , \quad dx^4 = idt \ , \qquad (2.15)$$

wherein by the analogy to the pseudo-real representation $i\beta''_{AB}$ is the 'supplementary' angle of $i\beta'_{AB}$. Thus, the Kerr metric has a simple explanation. The cross term does not stand for the properties of a Riemannian space, but has its origin in the implementation of an additional structure in this space. The rotational effects of the Kerr metric are invoked by locally rotated and stretched vectors.

3. MORE TRANSFORMATIONS

The Carter 4-bein system, which we call the system C, is a preferred reference system for the Kerr metric. Iyer and Kumar [12] used an other system, which we call the system A. It can easily be derived from the system C by a Lorentz transformation

$$\overset{m}{e}_i(A) = L^{m}_{m'}\overset{m'}{e}_i(C), \quad L^{3}_{3'} = \alpha_{AC}, \quad L^{3}_{4'} = -i\alpha_{AC}\omega_{AC}\sigma, \quad L^{4}_{3'} = i\alpha_{AC}\omega_{AC}\sigma, \quad L^{4}_{4'} = \alpha_{AC} \ . \tag{3.1}$$

Applying this transformation to (2.13) and suppressing the first two dimensions we obtain

$$\begin{aligned} &\overset{m}{e}_3 \doteq \underset{3}{h}^m = \{\cos i\beta_{AB}, \sin i\beta_{AB}\}\tau_3, \quad \overset{m}{e}_4 \doteq \underset{4}{h}^m = \{0, 1\}\tau_4 \\ &\underset{m}{e}^3 \doteq \overset{3}{h}_m = \{1,0\}\gamma^3, \quad \underset{m}{e}^4 \doteq \overset{4}{h}_m = \{-\sin i\beta_{AB}, \cos i\beta_{AB}\}\gamma^4 \end{aligned} \ . \tag{3.2}$$

A further system was introduced be Bardeen [13], we call it the system B. We derive it from the system C by

$$\overset{m}{e}_i(B) = L^{m}_{m'}\overset{m'}{e}_i(C), \quad L^{3}_{3'} = \alpha_{BC}, \quad L^{3}_{4'} = -i\alpha_{BC}\omega_{BC}\sigma, \quad L^{4}_{3'} = i\alpha_{BC}\omega_{BC}\sigma, \quad L^{4}_{4'} = \alpha_{BC} \tag{3.3}$$

and we obtain

$$\begin{aligned} &\overset{m}{e}_3 \doteq \underset{3}{h}^m = \{1,0\}\tau_3, \quad \overset{m}{e}_4 \doteq \underset{4}{h}^m = \{\sin i\beta_{AB}, \cos i\beta_{AB}\}\tau_4 \\ &\underset{m}{e}^3 \doteq \overset{3}{h}_m = \{\cos i\beta_{AB}, -\sin i\beta_{AB}\}\gamma^3, \quad \underset{m}{e}^4 \doteq \overset{4}{h}_m = \{0,1\}\gamma^4 \end{aligned} \ . \tag{3.4}$$

The system A is adjusted in such a way that the direction of the fourth tangent vector of A coincides with the direction of the fourth tangent vector of the oblique-angled system, while the third tangent vector of A is chosen to be normal to the other bein vectors. Its direction differs from the direction of the corresponding vector of the oblique-angled system by an angle $i\beta_{AB}$. For Bardeen's system we have to rotate the fourth tangent vector by $i\beta_{AB}$, while the direction of the third tangent vectors of B and the corresponding oblique-angled vectors coincide.

These three preferred reference systems have interesting properties. If the $u_m = \{0,0,0,1\}$ are the components of the four-velocity of an observer related to the system A we find the observer's motion to be free of shears

$$u_{(\alpha\,||\beta)} = 0, \quad \alpha,\beta = 1,2,3 \ . \tag{3.5}$$

Bardeen's observers are irrotational but they have shears

$$u_{[\alpha\,||\beta]} = 0 \ . \tag{3.6}$$

This system is also called locally non rotating system. The system C exhibits both kinds of motion

$$u_{\alpha\,||\beta} = u_{[\alpha\,||\beta]} + u_{(\alpha\,||\beta)} \ . \tag{3.7}$$

The first part of the formula means the Coriolis forces caused by the dragging effects of the rotating source, the second part means the shears of the motion having their origin in the differential rotation law. The components of the oblique-angled system are measured with respect to the orthogonal system C. We ask for the meaning of this system. In a former paper [14] we have discussed the static seed metric

$$ds^2 = \alpha_S^2 a_R^2 dr^2 + \Lambda^2 d\vartheta^2 + \sigma^2 d\varphi^2 + a_S^2 dx^{4^2}, \ \ dx^4 = idt \ , \tag{3.8}$$

which describes the fundamental properties of the geometry and provides a representation of the theory as a surface embedded in a higher dimensional flat space. An anholonomic non-Lorentzian transformation implements an additional structure on this surface, the oblique-angled system which is responsible for the rotational effects of the Kerr metric. The original orthogonal reference system of the seed metric remains as system C of the Kerr metric. Suppressing the first two dimensions again we read from (3.8) the bein vectors

$$\overset{3}{e}_{3'} = \sigma, \quad \overset{4}{e}_{4'} = a_S, \quad \underset{3}{e}^{3'} = \frac{1}{\sigma}, \quad \underset{4}{e}^{4'} = \frac{1}{a_S} \ . \tag{3.9}$$

Their lengths are

$$\tau_{3'} = \sigma, \quad \tau_{4'} = a_S, \quad \gamma^{3'} = \frac{1}{\sigma}, \quad \gamma^{4'} = \frac{1}{a_S} \ . \tag{3.10}$$

The transformation to the Kerr system C is performed with (no summation over k, k')

$$\begin{gathered} \underset{i}{h}^{m} = \Lambda_i^{i'} \underset{i'}{h}^{m}, \quad \overset{i}{h}_{m} = \Lambda_{i'}^{i} \overset{i'}{h}_{m}, \quad \Lambda_k^{i'} = \overline{\Lambda}_k^{i'} \tau_k \gamma^{i'}, \quad \Lambda_{k'}^{i} = \overline{\Lambda}_{k'}^{i} \tau_{k'} \gamma^{i'} \\ \overline{\Lambda}_k^{i'} = \begin{pmatrix} \alpha_{BC} & i\alpha_{AC}\omega_{AC}\sigma \\ -i\alpha_{BC}\omega_{BC}\sigma & \alpha_{AC} \end{pmatrix}, \quad \overline{\Lambda}_{k'}^{i} = \begin{pmatrix} \alpha_{AC} & -i\alpha_{AC}\omega_{AC}\sigma \\ i\alpha_{BC}\omega_{BC}\sigma & \alpha_{BC} \end{pmatrix} \ . \end{gathered} \tag{3.11}$$

With (2.7), (2.8), (2.9) we obtain the Kerr metric (2.1) again. It is sufficient to remark that the same procedure is applicable to the metric of the Kerr interior solution [15,16].

4. RELATED PROBLEMS

Reference systems of types A and B occur frequently in the literature. The system A is of Gödel type [16,17], and the world lines are helical. For slow rotations one has

$$\begin{aligned} d\overset{3}{x} &= \overset{3}{e}_i dx^i = \alpha_{AB}\tau_3 dx^3 \approx \sigma d\varphi \\ d\overset{4}{x} &= \overset{4}{e}_i dx^i = i\alpha_{AB}\omega_{AB}\sigma\tau_3 dx^3 + \tau_4 dx^4 \approx i\left[\omega_{AB}\sigma\cdot\sigma d\varphi + a_S dt\right] \end{aligned} \tag{4.1}$$

where $d\overset{3}{x}$ and $d\overset{4}{x}$ are quantities measured by the observers with their rods and clocks. For slow rotations the tetrad system B has the form

$$\begin{aligned} d\overset{3}{x} &= \overset{3}{e}_i dx^i = \tau_3 dx^3 + i\alpha_{AB}\omega_{AB}\sigma\tau_4 dx^4 \approx \sigma\left[d\varphi - \omega_{AB}a_S dt\right] \\ d\overset{4}{x} &= \overset{4}{e}_i dx^i = \alpha_{AB}\tau_4 dx^4 \approx i a_S dt \end{aligned} \tag{4.2}$$

and has approximately the structure of a Galilean transformation. In a nostalgic notation we write for the system A

$$\begin{aligned} &\overset{4}{e}_3 = \frac{g_{34}}{g_{44}}, \quad \overset{4}{e}_4 = \sqrt{g_{44}}, \quad {}'g_{33} = g_{33} - \overset{4}{e}_3\overset{4}{e}_3 \\ &\left(d\overset{3}{x}\right)^2 = \left(g_{33} - \frac{g_{34}g_{34}}{g_{44}}\right) dx^3 dx^3 \end{aligned} \quad . \tag{4.3}$$

This is the Landau-Lifshiz formula for infinitesimally neighbored points on the φ-circles. In a similar way one obtains for the system B

$$\begin{aligned} &\overset{3}{e}_3 = \sqrt{g_{33}}, \quad \overset{3}{e}_4 = \frac{g_{34}}{g_{33}}, \quad {}'g_{44} = g_{44} - \overset{3}{e}_4\overset{3}{e}_4 \\ &\left(d\overset{4}{x}\right)^2 = \left(g_{44} - \frac{g_{34}g_{34}}{g_{33}}\right) dx^4 dx^4 \end{aligned} \quad , \tag{4.4}$$

wherein $d\overset{4}{x}$ stands for the physical time interval. Unfortunately the Landau-Lifshiz formula is mostly considered in the literature also for cases where the decomposition (4.4) of the metric is more natural. In recent papers [18] the decompositions of stationary metrics into system A and B are also called threading and slicing.

5. CONCLUSIONS

We have been able to show that the Kerr metric can be reduced to a very simple form by the use of basic properties of the tangent and gradient vectors of the tetrad system. The rotational effects of the Kerr metric are implemented by the structure of an oblique-angled tangent and gradient vector system. Moreover, we have shown that the transition to other reference systems known in the literature can be performed by Lorentz transformations.

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