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FALL TIME IN THE SCHWARZSCHILD FIELD

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Contents

In previous papers we have shown that an observer falling in from an arbitrary position in the Schwarzschild field can only reach the event horizon in an infinite proper time. Since this result is in contradiction to the literature we will search for the reason.

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1. INTRODUCTION

Misner, Thorne, and Wheeler (MTW) [1] have extensively discussed the problem of free fall in their textbook 'Gravitation'. Their proposed solution has entered into some more textbooks and into more than a hundred publications. The derivation of the free fall from an arbitrary position takes several pages of a few sub-sections of the textbook.

In previous papers [2 - 5] we have discussed the rise and fall time of an observer in the Schwarzschild field by using standard Schwarzschild co-ordinates, Einstein-Rosen co-ordinates, and isotropic co-ordinates and also the angle of ascent of the Schwarzschild parabola and the rapidity of the motion as parameters. It has turned out that any observer infalling from the infinite or from an arbitrary position can only reach the event horizon in infinite proper time. Since this result contradicts the widespread opinion of the physicist community, we will thoroughly compare the derivation of MTW with our access to the problem.

2. DERIVATION BY MTW

First, we will repeat the derivation of the proper time of a freely falling object from an arbitrary position as done by MTW, but we will simplify it considerably. Thus, we will get more insight, but we will find inconsistencies.

In addition to the proper time τ the variables $\lambda = \tau/\mu$ with μ as the rest mass, the energy E at infinity, and the local energy E_{local} are used by MTW. Furthermore, the vectors are represented in the co-ordinate notation and as 1-forms as well. If a quantity is formulated with the help of a reference system, the indices are suppressed. Likewise the t-notation ($x^0 = t$) is used instead of the more convenient it-notation ($x^4 = it$). The t-notation requires a careful treatment of the time-like components of 4-vectors.

In MTW p 663 we find the relation

$$r_0 = \frac{2M}{1-\tilde{E}^2} \tag{2.1}$$

which they have educed after lengthy deliberations. Therein r_0 is the position where the observer has zero velocity (apastron), $\tilde{E} = E/m$ the energy per rest mass and E is the energy at infinity. Since we want to free ourselves from terms like 'energy at infinity', we have to rewrite Eq. (2.1) in such a way that it contains basic variables which are directly related to the free fall.

Since r_0 is the very position from which the observer (we will call him B') is released for the free fall, he has at this location the initial velocity $v' = 0$. A second observer (we will call him B''), who is in free fall coming from infinity has, at the moment he passes this position, the speed

$$v_0 = -\sqrt{\frac{2M}{r_0}} \quad . \tag{2.2}$$

By rearranging the equation (2.1) one gains

$$\tilde{E}^2 = 1 - v_0^2 = \frac{1}{\alpha_0^2} \quad .$$

Thus, one has

$$\tilde{E} = \frac{1}{\alpha_0} \quad , \tag{2.3}$$

wherein α_0 is the Lorentz factor for the velocity of the observer B'' who comes from infinity and is just passing the position r_0.

MTW start with their considerations from the equation (p 656)

$$g_{\alpha\beta} p^\alpha p^\beta + m^2 = 0 \quad , \tag{2.4}$$

wherein p^α is the 4-momentum. Going on with the tetrad representation and with the it-notation, and by dividing by m^2 and by multiplying by the proper time, we arrive at a relation that can be derived from the invariance of the line element with respect to Lorentz transformation. Reduced to two dimensions, this is

$$ds^2 = dx'^2 - dT'^2 = dx^2 - dT^2 \quad , \tag{2.5}$$

wherein dx' and dT' are the physical radial and time-like arc elements of the metric in terms of an observer B' who is infalling from r_0. dx and dT refer to the static observer in the Schwarzschild field (we will call him B). For this observer the relations

$$dx = \alpha dr, \quad dT = \frac{1}{\alpha} dt, \quad \alpha = \frac{1}{\sqrt{1 - \frac{2M}{r}}}$$

are to be applied, where the metric coefficient α is identical with the Lorentz factor of an observer B'' incoming from the infinite.

In a reference system that is linked to an observer B' falling down from r_0, one has

$$x' = \text{const.}, \quad dx' = 0 \quad . \tag{2.6}$$

If one utilizes these definitions one obtains from (2.5)

$$-dT'^2 = dx^2 - dT^2 \quad , \tag{2.7}$$

which corresponds to the MTW relation (2.4), if one undoes all the changes in notation. Clearly, (2.7) is more insightful than (2.4). We write (2.7) in the form

$$dx^2 = \left(\frac{dT^2}{dT'^2} - 1 \right) dT'^2 \quad . \tag{2.8}$$

From now on we perform the calculations in such a way that we get the result of MTW. We repeatedly use the formulae of the table beneath.

I. $x'' = \text{const.}$

systems	L	transformations	rel. velocities	phys. time	rel. vel. of	meas. in
1. B'' || B'	L(v_0)	$dx^{m''} = L^{m''}_{m'} dx^{m'}$	$\frac{dx'}{dT'} = v_0$	$\frac{dT'}{dT''} = \alpha_0$	B'' a. B'	B'
2. B'' || B	L(v)	$dx^{m''} = L^{m''}_{m} dx^{m}$	$\frac{dx}{dT} = v$	$\frac{dT}{dT''} = \alpha$	B'' a. B	B
3. B || B'	L(v')	$dx^{m} = L^{m}_{m'} dx^{m'}$	$\frac{dx''}{dT''} = 0$	$\frac{dT}{dT'} = \frac{\alpha}{\alpha_0}$		

II. $x' = \text{const.}$

systems	L	transformations	rel. velocities	phys. time	rel. vel. of	meas. in
1. B' || B	L(v')	$dx^{m'} = L^{m'}_{m} dx^{m}$	$\frac{dx}{dT} = v'$	$\frac{dT}{dT'} = \alpha'$	B' a. B	B
2. B' || B''	L(v_0)	$dx^{m'} = L^{m'}_{m''} dx^{m''}$	$\frac{dx''}{dT''} = -v_0$	$\frac{dT''}{dT'} = \alpha_0$	B' a. B''	B''
3. B'' || B	L(v)	$dx^{m''} = L^{m''}_{m} dx^{m}$	$\frac{dx'}{dT'} = 0$	$\frac{dT''}{dT} = \frac{\alpha_0}{\alpha'}$		

III. $x = \text{const.}$

systems	L	transformations	rel. velocities	phys. time	rel. vel. of	meas. in
1. B || B'	L(v')	$dx^{m} = L^{m}_{m'} dx^{m'}$	$\frac{dx'}{dT'} = -v'$	$\frac{dT'}{dT} = \alpha'$	B a. B'	B'
2. B || B''	L(v)	$dx^{m} = L^{m}_{m''} dx^{m''}$	$\frac{dx''}{dT''} = -v$	$\frac{dT''}{dT} = \alpha$	B a. B''	B''
3. B' || B''	L(v_0)	$dx^{m'} = L^{m'}_{m''} dx^{m''}$	$\frac{dx}{dT} = 0$	$\frac{dT'}{dT''} = \frac{\alpha'}{\alpha}$		

In (2.8) we are using

$$\frac{dT}{dT'} = \frac{\alpha}{\alpha_0}, \quad \alpha_0 = \frac{1}{\sqrt{1-v_0^2}}, \quad v_0 = -\sqrt{\frac{2M}{r_0}} \ , \tag{2.9}$$

and we will reflect the first relation therein later on. Substituting the first relation (2.9) into (2.8) one obtains

$$dx^2 = \left(\frac{\alpha^2}{\alpha_0^2} - 1\right) dT'^2 = \alpha^2 \left(\frac{1}{\alpha_0^2} - \frac{1}{\alpha^2}\right) dT'^2 \quad .$$

$$dr^2 = \left[\left(1 - v_0^2\right) - \left(1 - v^2\right)\right] dT'^2 = \left(v^2 - v_0^2\right) dT'^2$$

Finally, we have for the proper time of an observer B' who falls down from r_0

$$dT' = \frac{1}{\sqrt{v^2 - v_0^2}} dr \quad . \tag{2.10}$$

By using the standard Schwarzschild metric coefficients as in $dT = \frac{1}{\alpha} dt$ we get with

$$dT' = \frac{\alpha_0}{\alpha} dT = \frac{\alpha_0}{\alpha^2} dt \tag{2.11}$$

an expression for the co-ordinate time related to the static observer

$$dt = \sqrt{\frac{1 - v_0^2}{v^2 - v_0^2}} \frac{1}{1 - v^2} dr \quad . \tag{2.12}$$

For the integral of (2.12) MTW provide a solution with cycloid parameters. Until now we have carried out a method, how to get, under simplified conditions, the result of MTW and we will be able refer to these statements hereafter.

MTW do not explicitly provide a formula for the fall velocity of an observer who is released from r_0. But it can be found in the textbook by Raine and Thomas [6] and can be derived with

$$\frac{dx}{dt} = v' \quad ,$$

where we have taken the above expression from the table for $x' = \text{const.}$. With $dx = \alpha dr,\ dT = \frac{1}{\alpha} dt$ we finally obtain with the MTW formula (2.10)

$$v' = \sqrt{\frac{v^2 - v_0^2}{1 - v_0^2}} \quad . \tag{2.13}$$

We recognize that this formula contradicts the Einstein addition law of velocities. If one represents this function graphically for some values of r_0, one can see that the velocity of freely falling objects v' reaches the speed of light at the event horizon for all r_0. However, the curves alter the curvatures at certain points. It is not very plausible that Nature provides for the free fall preferred points, in which the velocity changes significantly its behavior. We illustrate this in Fig. 2.1.

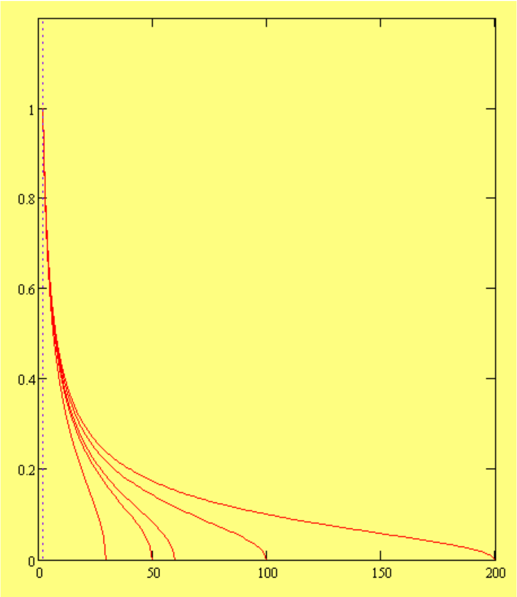

Fig. 2.1

3. CORRECTING THE DERIVATION

Previously, we have noted that we are reserved concerning the use of the first equation (2.9). We will now examine this relation in more detail. From the table we see that this relation is valid for $x'' = \text{const.}$, i.e. for an observer B", who is associated with a system that comes from infinity. We want to reconsider this once more in detail.

We note the Lorentz transformations

$$\begin{aligned} &L_1^{1'} = \alpha', \quad L_1^{4'} = -i\alpha' v', \quad L_4^{1'} = i\alpha' v', \quad L_4^{4'} = \alpha' \\ &L_{1'}^{1''} = \alpha_0, \quad L_{1'}^{4''} = -i\alpha_0 v_0, \quad L_{4'}^{1''} = i\alpha_0 v_0, \quad L_{4'}^{4''} = \alpha_0 \;, \\ &L_1^{1''} = \alpha, \quad L_1^{4''} = -i\alpha v, \quad L_4^{1''} = i\alpha v, \quad L_4^{4''} = \alpha \end{aligned} \tag{3.1}$$

relating the three observers B, B', and B" and we also note the Lorentz formulae

$$v' = \frac{v - v_0}{1 - v v_0}, \qquad v = \frac{v' + v_0}{1 + v' v_0}, \qquad v_0 = \frac{v - v'}{1 - v v'} \;, \tag{3.2}$$

$$\alpha' = \alpha\alpha_0 (1 - v v_0), \quad \alpha = \alpha'\alpha_0 (1 + v' v_0), \quad \alpha_0 = \alpha'\alpha (1 - v' v) \;, \tag{3.3}$$

$$\alpha' v' = \alpha\alpha_0 (v - v_0), \quad \alpha v = \alpha'\alpha_0 (v' + v_0), \quad \alpha_0 v_0 = \alpha\alpha' (v - v') \;. \tag{3.4}$$

With the help of (3.1) one obtains

$$dx^{4'} = -i\alpha' v' dx^1 + \alpha' dx^4 \,,$$

and with $dx^{4'} = idT'$, $dx^4 = idT$

$$dT' = \alpha' \left[1 - v' \frac{dx}{dT} \right] dT \ .$$

Since the velocity of B" with respect to the static system B is defined by $v = \frac{dx}{dT}$, one has finally derived

$$\frac{dT}{dT'} = \frac{\alpha}{\alpha_0}$$

with the help of (3.3), last relation. We still have to show that this applies to $x'' = \text{const.}$. With the Lorentz transformation (3.1) holds for $x'' = \text{const.}$ the relation

$$dx^{1''} = \alpha dx^1 + i\alpha v dx^4 = 0 \ .$$

This yields the aforementioned expression for the relative velocity

$$v = \frac{dx}{dT} \ ,$$

as stipulated in the table. We have performed quite elementary calculations to show which formulae are valid if one associates a reference system with a moving observer.

We now recognize the problems of the MTW method. The relation (2.7) was derived by applying $x' = \text{const.}$. Then an expression was used which refers to a relation with $x'' = \text{const.}$. That means that two excluding conditions are used in the same equation.

If we take from the table the appropriate expression for $x' = \text{const.}$, namely

$$\frac{dT}{dT'} = \alpha' \tag{3.5}$$

we refer to a single reference system, namely to the one which is associated with the observer B' who is released from r_0. Substituting in (2.8) leads after a short calculation to

$$dT' = \frac{\alpha}{\alpha' v'} dr \ ,$$

which can be directly derived from

$$\frac{dx^1}{dT} = v', \quad \frac{\alpha dr}{dT'} = \alpha' v' \ . \tag{3.6}$$

With the Lorentz formula of (3.4) we get

$$dT' = \frac{\alpha}{\alpha' v'} dr = \frac{1}{\alpha_0 \left(v - v_0 \right)} dr \ . \tag{3.7}$$

For the question which of the two approaches is correct, there exists a decisive criterion. According to Einstein no gravity acts on objects which are inside a freely falling elevator. They have to hover in the elevator, independently of the position from which the elevator starts. In a former paper [4] we have shown that this is the case for our approach, and therefore the method of MTW has to be dismissed.

4. REFERENCES

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