ARG Austrian Reports on Gravitation

ARG-2017-2

REMARKS ON THE COSMOLOGICAL MODEL OF MILNE

Rainer Burghardt*

Keywords: cosmology, model of Milne, coordinate transformation, space curvature

Abstract: The position often communicated in the literature that the curvature properties of an expanding cosmological model might depend on the representation in certain coordinates is not generally accepted. Using the model of Milne, we show that the scale factor of the metric is confused with the time variable.

In 1934 Milne [1,2] has published a cosmological model based on the principles of special relativity. In a flat space there is first concentrated matter, which spreads into all directions after an initial process. The redshift of the light emanating from the receding stars has a kinematic origin and is explained by the Doppler effect. Although the model of Milne has been rejected for physical reasons, some cosmologists still have some interest [3-9] in it.

The Milne cosmos is also regarded as a special case of the Friedman cosmos with the curvature parameter being $k=-1$. With a special coordinate transformation one wants to bring the metric into a flat form in order to obtain the original description of the Milne cosmos. We want to highlight this process critically.

We carry on from the metric of the Friedman cosmos in case of negative constant spatial curvature

$$ds^2 = \mathcal{R}^2\left[d\eta^2 + sh^2\eta d\vartheta^2 + sh^2\eta \sin^2\vartheta d\varphi^2\right] - dt^2, \quad \mathcal{R} = \mathcal{R}(t). \tag{1}$$

With the radial coordinate $r = \mathcal{R}\,sh\eta$ we obtain the metric in canonical form

$$ds^2 = \frac{1}{1+\frac{r^2}{\mathcal{R}^2}} dr'^2 + r^2 d\vartheta^2 + r^2 \sin^2\vartheta d\varphi^2 - dt^2 \tag{2}$$

and read from it $k=-1$. From (1) we borrow the 4-bein system:

* e-mail: arg@aon.at, home page: http://arg.or.at/

$$\overset{1}{e}_1 = \mathcal{R}, \quad \overset{2}{e}_2 = \mathcal{R}\,\mathrm{sh}\eta, \quad \overset{3}{e}_3 = \mathcal{R}\,\mathrm{sh}\eta \sin\vartheta, \quad \overset{4}{e}_4 = 1. \tag{3}$$

From this, we calculate the field strengths from the Ricci-rotation coefficients [11]

$$A_{mn}{}^{s} = \overset{s}{e}_i \, e^i_{[n|m]} + g^{sr} g_{mt} \overset{t}{e}_i \, e^i_{[n|r]} + g^{sr} g_{nt} \overset{t}{e}_i \, e^i_{[m|r]}. \tag{4}$$

We separate them into

$$A_{mn}{}^{s} = B_{mn}{}^{s} + C_{mn}{}^{s} + U_{mn}{}^{s} \tag{5}$$

and with the unit vectors

$$m_m = \{1,0,0,0\}, \quad b_m = \{0,1,0,0\}, \quad c_m = \{0,0,1,0\}, \quad u_m = \{0,0,0,1\} \tag{6}$$

we split up further into

$$B_{mn}{}^{s} = b_m B_n b^s - b_m b_n B^s, \quad C_{mn}{}^{s} = c_m C_n c^s - c_m c_n C^s, \quad U_{mn}{}^{s} = u_m U_n u^s - u_m u_n U^s. \tag{7}$$

B and C are the lateral field quantities and U is the radial field quantity. For the 4th components of all field quantities we get

$$U_4 \overset{*}{=} B_4 \overset{*}{=} C_4 = \frac{1}{\mathcal{R}} \mathcal{R}_{|4}. \tag{8}$$

This means that the expansions in all three spatial directions are the same. Therefore, the curvature scalar has the form

$$u^s{}_{||s} = \frac{3}{\mathcal{R}} \mathcal{R}_{|4}.$$

The model expands in free fall because $g_{44} = 1$ is the metric coefficient in (1). Thus, the proper time T is equal to the coordinate time t. The right-hand side of (8) can also be written as

$$\frac{i}{\mathcal{R}} \frac{\partial \mathcal{R}}{\partial T} = \frac{i}{\mathcal{R}} \mathcal{R}^{\bullet}. \tag{9}$$

Furthermore, we put with regard to Milne's ansatz

$$\mathcal{R}^{\bullet} = 1, \tag{10}$$

where 1 is the velocity of light in the natural measuring system. Finally, with (4)-(8) one has

$$\begin{aligned}
U_m &= A_{1m}{}^{1} = \{0,0,0,1\}\left(-\frac{i}{\mathcal{R}}\right) \\
B_m &= A_{2m}{}^{2} = \left\{\frac{1}{\mathcal{R}} \mathrm{cth}\eta, 0, 0, -\frac{i}{\mathcal{R}}\right\} = \{\mathrm{ch}\eta, 0, 0, -i\,\mathrm{sh}\eta\}\left(\frac{1}{\mathcal{R}\,\mathrm{sh}\eta}\right) \\
C_m &== A_{3m}{}^{3} = \left\{\frac{1}{\mathcal{R}} \mathrm{cth}\eta, \frac{1}{\mathcal{R}\,\mathrm{sh}\eta} \cot\vartheta, 0, -\frac{i}{\mathcal{R}}\right\} \\
&= \{\mathrm{ch}\eta \sin\vartheta, \cos\vartheta, 0, -i\,\mathrm{sh}\eta \sin\vartheta\}\left(\frac{1}{\mathcal{R}\,\mathrm{sh}\eta \sin\vartheta}\right)
\end{aligned}. \tag{11}$$

Therein $\frac{i}{\mathcal{R}},\frac{1}{\mathcal{R}\,\mathrm{sh}\eta},\frac{1}{\mathcal{R}\,\mathrm{sh}\eta\sin\vartheta}$ are the curvatures of normal and oblique slices of the surface with the metric (1), which is the base of the model. The relations

$$\underset{1}{U^{s}_{||s}}+U^{s}U_{s}=0,\quad \underset{2}{B_{m||n}}+B_{m}B_{n}=0,\quad \underset{3}{C_{m||n}}+C_{m}C_{n}=0 \tag{12}$$

are formulated with the unit vectors (6) and the graded derivatives

$$\begin{gathered}\underset{1}{U_{m||n}}=U_{m|n},\quad \underset{2}{B_{m||n}}=B_{m|n}-U_{nm}{}^{s}B_{s},\quad \underset{2}{C_{m||n}}=C_{m|n}-U_{nm}{}^{s}C_{s}-B_{nm}{}^{s}C_{s}\\ U_{nm}{}^{s}=u_{n}U_{m}u^{s}-u_{n}u_{m}U^{s},\quad B_{nm}{}^{s}=b_{n}B_{m}b^{s}-b_{n}b_{m}B^{s}\end{gathered}. \tag{13}$$

They are subequations of the Ricci [11]. Since all the expressions in (12) vanish one has $R_{mn}=0$. Therefore the stress-energy-momentum tensor is $T_{mn}=0$. The universe is empty.

The special feature of the Milne model is the simple dependency of the scale factor on time:

$$\mathcal{R}(t)=t\,. \tag{14}$$

Instead of (9) one has

$$\frac{i}{t}\frac{\partial t}{\partial T}=\frac{i}{t}\frac{\partial t}{\partial t}=\frac{i}{t}$$

and instead of (10) the trivial relation $t^{\bullet}=1$ which justifies the approach (10) in retrospect.

With this and with $r=\mathcal{R}\,\mathrm{sh}\eta$ the simple expressions for the field quantities are obtained

$$U_{m}=\{0,0,0,1\}\left(-\frac{i}{t}\right),\quad B_{m}=\left\{\frac{1}{r}\mathrm{ch}\eta,0,0,-\frac{i}{t}\right\},\quad C_{m}=\left\{\frac{1}{r}\mathrm{ch}\eta,\frac{1}{r}\cot\vartheta,0,-\frac{i}{t}\right\}. \tag{15}$$

If we also identify the quantity t introduced by (14) with the time variable, it is possible to apply the coordinate transformation

$$r'=t\,\mathrm{sh}\eta,\quad t'=t\,\mathrm{ch}\eta\,. \tag{16}$$

Thus, one first obtains

$$dr'^{2}-dt'^{2}=t^{2}d\eta^{2}-dt^{2}.$$

Now the metric (1) can be written as

$$ds^{2}=t^{2}\left[d\eta^{2}+\mathrm{sh}^{2}\eta\,d\vartheta^{2}+\mathrm{sh}^{2}\eta\sin^{2}\vartheta\,d\varphi^{2}\right]-dt^{2} \tag{17}$$

and with the above relation as

$$\begin{gathered}ds^{2}=dr'^{2}+r'^{2}d\vartheta^{2}+r'^{2}\sin^{2}\vartheta\,d\varphi^{2}-dt'^{2}\\ \overset{1'}{e}_{1'}=1,\quad \overset{2'}{e}_{2'}=r',\quad \overset{3'}{e}_{3'}=r'\sin\vartheta,\quad \overset{4'}{e}_{4'}=1\end{gathered}. \tag{18}$$

This is obviously the metric in the flat Minkowski space. It seems that with a coordinate transformation

$$\Lambda_{i}^{i'}=\begin{pmatrix}t\,\mathrm{ch}\eta & -i\,\mathrm{sh}\eta\\ t\,i\,\mathrm{sh}\eta & \mathrm{ch}\eta\end{pmatrix}$$

and the associated pseudo rotation

$$L_m^{m'} = \begin{pmatrix} ch\eta & -ish\eta \\ ish\eta & ch\eta \end{pmatrix} = \begin{pmatrix} \cos i\eta & -\sin i\eta \\ \sin i\eta & \cos i\eta \end{pmatrix}, \quad L_m^{m'} = \overset{m'}{e}_{i'} \Lambda_i^{i'} \underset{m}{e}^i \tag{19}$$

a negatively curved space $(k = -1)$ can be made from a flat space $(k = 0)$.

With the help of the pseudo rotation (19) the field quantities (11) can be converted into those of a static system. The Ricci-rotation coefficients transform according to

$$'A_{m'n'}{}^{s'} = L_{m'n's}^{m\,n\,s'} A_{mn}{}^{s} + 'L_{m'n'}{}^{s'}.$$

The quantity U is the very component of the Ricci-rotation coefficients, which transforms inhomogeneously

$$'U_{m'n'}{}^{s'} = U_{m'n'}{}^{s'} + 'L_{m'n'}{}^{s'}.$$

With

$$U_{m'n'}{}^{s'} = h_m{}^{s'} U_{n'} - h_{m'n'} U^{s'}, \quad h_{m'n'} = \mathrm{diag}\{1,0,0,1\}$$
$$'L_{m'n'}{}^{s'} = h_{m'}{}^{s'}\, 'L_{n'} - h_{m'n'}\, 'L^{s'}, \quad 'L_{n'} = 'L_{s'n'}{}^{s'} = \{'L_{4'1'}{}^{4'}, 'L_{1'4'}{}^{1'}\} \tag{20}$$

one finally obtains the simple relations

$$'U_{m'} = U_{m'} + 'L_{m'}\ .$$

With

$$'L_{m'n'}{}^{s'} = L_s^{s'} L_{n'|m'}^{s}, \quad 'L_{1'} = 'L_{4'1'}{}^{4'} = -i\eta_{|4'}, \quad 'L_{4'} = 'L_{1'4'}{}^{1'} = i\eta_{|1'}, \tag{21}$$

as well as with $\eta_{|m'} = L_{m'}^{m} \underset{m}{e}^i \eta_{|i}$ and (3) one first has

$$'L_{m'} = \{-ish\eta, 0, 0, ch\eta\} \frac{i}{t} \tag{22}$$

and from the inhomogeneous transformation law of the Ricci-rotation coefficients

$$U_{m'} = L_{m'}^{m} U_m = \{-ish\eta, 0, 0, ch\eta\}\left(-\frac{i}{t}\right), \quad 'U_{m'} = U_{m'} + 'L_{m'} = 0\,. \tag{23}$$

In the transformed system one does not experience expansion, the space would be not only flat but also static.

In order to clarify this contradiction, we once again deal with the transformation (16) and we remember Eq. (14). Thus, we obtain

$$r' = \mathcal{R} sh\eta, \quad t' = \mathcal{R} ch\eta \tag{24}$$

and we recognize that t' is an inappropriate identification with the time variable. Instead we write

$$x^{1'} = \mathcal{R} sh\eta, \quad x^{0'} = i\mathcal{R} ch\eta \tag{25}$$

and we obtain with

$$dx^{1'} = sh\eta d\mathcal{R} + \mathcal{R} ch\eta d\eta, \quad dx^{0'} = ich\eta d\mathcal{R} + i\mathcal{R} sh\eta d\eta \tag{26}$$

after all

$$dx^{0'2} + dx^{1'2} = (id\mathcal{R})^2 + \mathcal{R}^2 d\eta^2\,. \tag{27}$$

Now one can see that the angle $i\eta$ describes a rotation in the [0',1']-plane of the embedding space and not a rotation in the [1',4']-plane of the physical space. Therefore (19) cannot be a Lorentz transformation which performs the transition from the comoving to the non-comoving system. (25) describes the relation between the hyperspheric coordinates of the Milne model and the Cartesian coordinates of the flat 5-dimensional embedding space. The complete transformation reads as

$$\begin{aligned} x^{3'} &= i\mathfrak{R}\sin i\eta \sin\vartheta \sin\varphi = \mathfrak{R}\,\mathrm{sh}\eta \sin\vartheta \sin\varphi \\ x^{2'} &= i\mathfrak{R}\sin i\eta \sin\vartheta \cos\varphi = \mathfrak{R}\,\mathrm{sh}\eta \sin\vartheta \cos\varphi \\ x^{1'} &= i\mathfrak{R}\sin i\eta \cos\vartheta = \mathfrak{R}\,\mathrm{sh}\eta \cos\vartheta \\ x^{0'} &= i\mathfrak{R}\cos i\eta = i\mathfrak{R}\,\mathrm{ch}\eta \\ x^{4'} &= it \end{aligned} \quad . \tag{28}$$

The Milne model with the metric (1) can be expressed geometrically as the surface of a pseudo-sphere

$$x^{a'}x^{a'} = \left(i\mathfrak{R}\right)^2, \quad a' = 0',1',...,3'$$

with an imaginary radius. This corresponds to the tradition of treating cosmological models. Recalling the above-mentioned considerations we can see that one has obtained the flat metric (18). One has performed a transformation into the flat embedding space. The misleading approach (16) has possibly been used first by Walker [10] and has been adopted by other authors.

Conclusions

The attempt to transform the comoving coordinate system of the Milne universe into a non-comoving coordinate system has failed because the radius of curvature of the imaginary hypersphere defining the geometry has been confused with the time variable. However, we can reject the assertion that the curvature of the space may depend on the representation by pointing out these circumstances.

References

[1] Milne E. A., *A Newtonian expanding universe*. Quart. Journ. Math. Oxford **5**, 64, 1934
[2] Milne E. A., *World-structure and the expansion of universe.* Z. Astrophys. **6**, 1, 1933
[3] McCrea W. H., Milne E. A., *Newtonian universe and the curvature of space.* Quart. J. Math. Oxford. **5**, 73, 1934
[4] Layzer D., *On the significance of Newtonian cosmology*. Astr. Journ. **59**, 286, 1954
[5] McCrea W, H., *The significance of Newtonian cosmology*. Astr. Journ. **60**, 271, 1955
[6] Chodorowski M. J., *Cosmology under Milne's shado*w. astro-ph/0503690
[7] Macleod A., *An interpretation of Milne cosmology*. physics/051070
[8] Macleod A., *Evidence for a universe expanding at the speed of light.* physics/0407084
[9] Rindler W., *Public and private space curvature in Robertson-Walker universe*. GRG **13**, 457, 1981
[10] Walker A. G., *On Milne's theory of world-structure.* Proc. Lond. Math. Soc. **42**, 90, 1937
[11] Burghardt R., 2005, *Spacetime curvature*. http://members.wavenet.at/arg/EMono.htm
Raumkrümmung. http://members.wavenet.at/arg/Mono.htm