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A NUT INTERIOR

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Contents

We supplement the NUT metric with an interior solution and set up the field equations and the stress-energy tensor.

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1. INTRODUCTION

Although no evidence has been found that physical meaning can be ascribed to the NUT metric this metric is of mathematical interest, because it is a generalization of the Schwarzschild metric and is in close relation with the Reissner-Nordström metric. It is mathematically neat to supplement the NUT metric with an interior. The model we present fits in the Kerr family and is reduced to the Schwarzschild metric putting the NUT parameter zero.

According to our previous paper we set up the model in three steps. In Sec. 2 we fit a cap of a sphere to the surface of the exterior solution described in [1]. The geometry of this ansatz is analyzed in Sec. 3. In Sec. 4 we gauge the rods and clocks of the model. This leads us from the reduced metric to the seed metric. In Sec. 5 we gain the interior NUT with an intrinsic transformation invoking the rotational content of the model. We calculate the field strengths, the field equations, and the stress-energy tensor. The stress-energy tensor contains Schwarzschild-like contributions, and in addition, a monopole term similar to the magnetic monopole term sometimes discussed in electrodynamics.

2. BASICS OF THE NUT INTERIOR

The underlying geometrical object for the NUT interior is the cap of a sphere. The construction of this cap is analogous to the one of the Schwarzschild interior [2]. The $\mathcal{R}$ are the radii of a family of spheres. With the *embedding condition*

$$\mathcal{R} = \mathcal{R}_g = \text{const.} \tag{2.1}$$

we select from this family that cap which is suitable for matching the exterior solution. The polar angle is η and is related to the standard-Schwarzschild co-ordinate by

$$r = \mathcal{R}\sin\eta \ . \tag{2.2}$$

Thus, one has

$$\sin\eta = \frac{r}{\mathcal{R}}, \quad \cos\eta = \sqrt{1-\frac{r^2}{\mathcal{R}^2}} \ . \tag{2.3}$$

The *reduced metric* has the form

$$ds^2 = \mathcal{R}^2 d\eta^2 + \mathcal{R}^2\sin^2\eta\, d\vartheta^2 + \mathcal{R}^2\sin^2\eta\sin^2\vartheta\, d\varphi^2 + a_T^2 dit^2 \ . \tag{2.4}$$

With the help of (2.2) it can be written as

$$ds^2 = \frac{1}{1-\frac{r^2}{\mathcal{R}^2}} dr^2 + r^2 d\vartheta^2 + r^2 \sin^2\vartheta d\varphi^2 + a_T^2 dit^2 \ . \quad (2.5)$$

The time-factor of the metric is

$$a_T = a_T(\mathcal{R},\eta) = \left[\left(\mathcal{R}_g + \rho_g\right)\cos\eta_g - \mathcal{R}\cos\eta\right]\frac{1}{\rho_g} \ . \quad (2.6)$$

η_g is the aperture angle of the cap matching the interior to the exterior, r_g the position of the boundary surface, and ρ_g the radius of curvature of the radial curves of the exterior surface at the boundary surface

$$\rho_g = \frac{\sqrt{2Mr_g + l^2}}{Mr_g + l^2} r_g^2 \ . \quad (2.7)$$

It differs from the radius of curvature of the Schwarzschild solution by the NUT parameter l. Putting l zero, one obtains the corresponding value of the Schwarzschild geometry $\rho_g = \sqrt{2r_g^3/M}$. Interpreting the time interval by

$$dit = \rho_g di\psi \quad (2.8)$$

one obtains for the time-like part of the line element

$$dx^4 = \left[\left(\mathcal{R}_g + \rho_g\right)\cos\eta_g - \mathcal{R}\cos\eta\right] di\psi \ . \quad (2.9)$$

Actually (2.9) represents the line elements of two concentric pseudo circles within the framework of a double surface theory [2]. Defining the constant auxiliary quantity

$$2\Phi_g^2 = \frac{2Mr_g + l^2}{Mr_g + l^2} \quad (2.10)$$

one is able to establish a closer relation between the quantities ρ_g and $\mathcal{R}$

$$\rho_g = 2\mathcal{R}_g\Phi_g^2 = \mathcal{R}_g \frac{2Mr_g + l^2}{Mr_g + l^2} \quad (2.11)$$

and to write the time factor of the reduced metric in a Schwarzschild-like form

$$a_T = \frac{1}{2}\left[\left(1 + 2\Phi_g^2\right)\cos\eta_g - \cos\eta\right]\Phi_g^{-2} \quad (2.12)$$

by the use of the embedding condition (2.1). (2.11) describes the ratio of ρ_g and $\mathcal{R}_g$. From (2.10) it is obvious that one obtains for the Schwarzschild case $\Phi_g^2 = 1$. Thus, one gets the well-known Schwarzschild factor

$$a_T = \frac{1}{2}\left[3\cos\eta_g - \cos\eta\right]. \tag{2.13}$$

It should be noted that one can obtain from the reduced NUT metric our interior Reissner-Nordström metric [3] by putting l = ie. The design of the reduced NUT model is akin to the Reissner-Nordström interior. From (2.7) and (2.11) one derives the *junction condition*

$$\mathcal{R}_g = \frac{r_g^2}{\sqrt{2Mr_g + l^2}} \ . \tag{2.14}$$

If one has decided for the junction position r_g one can calculate the radius assigned to the selected cap.

From (2.4) we infer the operators

$$\partial_0 = \frac{\partial}{\partial\mathcal{R}}, \quad \partial_1 = \frac{\partial}{\mathcal{R}\partial\eta}, \quad \partial_2 = \frac{\partial}{\mathcal{R}\sin\eta\partial\vartheta}, \quad \partial_3 = \frac{\partial}{\mathcal{R}\sin\eta\sin\vartheta\partial\varphi}, \quad \partial_4 = \frac{\partial}{a_T\partial it} \tag{2.15}$$

that we need for the calculation of the field strengths. We emphasize that the time factor is a function of $\mathcal{R}$ and η. As long as one describes the full geometry, the extrinsic and the intrinsic, $\mathcal{R}$ has to be considered as a variable. Therefore, one has

$$a_{T|0} = -\frac{1}{\rho_g}\cos\eta, \quad a_{T|1} = \frac{1}{\rho_g}\sin\eta \ . \tag{2.16}$$

Now it is possible to calculate from the metric the 5-dimensional components of the field strengths (a = 0,1,…,4)

$$\begin{gathered} M_a = \left\{\frac{1}{\mathcal{R}},0,0,0,0\right\}, \quad B_a = \left\{\frac{1}{\mathcal{R}},\frac{1}{\mathcal{R}}\cot\eta,0,0,0\right\} \\ C_a = \left\{\frac{1}{\mathcal{R}},\frac{1}{\mathcal{R}}\cot\eta,\frac{1}{\mathcal{R}\sin\eta}\cot\vartheta,0,0\right\}, \quad E_a = \left\{\frac{1}{\rho_g a_T}\cos\eta,-\frac{1}{\rho_g a_T}\sin\eta,0,0,0\right\} \end{gathered} . \tag{2.17}$$

By the use of the 5-dimensional graded derivatives [1,2], one obtains with (2.17) the curvature equations

$$\begin{aligned} &M_{a\underset{1}{|||}b} + M_a M_b = 0, \quad M^c{}_{\underset{1}{|||}c} + M^c M_c = 0 \\ &B_{a\underset{2}{|||}b} + B_a B_b = 0, \quad B^c{}_{\underset{2}{|||}c} + B^c B_c = 0 \\ &C_{a\underset{3}{|||}b} + C_a C_b = 0, \quad C^c{}_{\underset{3}{|||}c} + C^c C_c = 0 \\ &E_{a\underset{4}{|||}b} - E_a E_b = 0, \quad E^c{}_{\underset{4}{|||}c} - E^c E_c = 0 \end{aligned} . \tag{2.18}$$

They are subequations of the identically vanishing 5-dimensional Ricci, which is the Ricci of the flat embedding space. With these quantities and relations, the basic structure of the interior NUT solution is entirely described.

3. THE REDUCED NUT METRIC

In the preceding Section, a metric akin to the Schwarzschild interior metric was introduced. The difference of those two metrics mainly appear in the time factor a_T. From the Schwarzschild metric [2] we know that a single surface is not sufficient for embedding the model into a 5-dimensional flat space. The theory of double surfaces has to be applied. Therefore, we present this problem in more detail. The reduced model can be deduced from a pseudo-hyper sphere in the same manner as the Schwarzschild model. We concentrate on the time-like part of the metric. The time-like element on the pseudo-hyper sphere

$$dX^4 = -\mathcal{R}\cos\eta\, d i\psi \tag{3.1}$$

is connected with the time-like element dx^4 of the *physical surface* by means of

$$dX^4 = \mathfrak{p}^4_4 dx^4 \ . \tag{3.2}$$

$\mathfrak{p} = \mathfrak{p}^4_4$ is a component of the projectors $\mathfrak{p}^a_b$, which transmutes the pseudo-hyper sphere into the NUT interior surface. Since dx^4 is known from (2.9) one can equate

$$-\mathcal{R}\cos\eta\, d i\psi = \mathfrak{p} a_T \rho_g d i\psi \ .$$

and can make accessible the negative[1] quantity

$$\mathfrak{p} = -\frac{\mathcal{R}\cos\eta}{\rho_g a_T} = -\frac{1}{\left(\frac{\mathcal{R}_g}{\mathcal{R}} + \frac{\rho_g}{\mathcal{R}}\right)\frac{\cos\eta_g}{\cos\eta} - 1} \ . \tag{3.3}$$

With this projector the components of the force of gravity are calculated from the corresponding components of the pseudo-spherical geometry

$$E_a = \mathfrak{p}\left\{-\frac{1}{\mathcal{R}}, \frac{1}{\mathcal{R}}\tan\eta, 0, 0, 0\right\} \ . \tag{3.4}$$

This projector makes it also possible to present the stress-energy tensor of the model in a plain form and to compare it with the Schwarzschild and Reissner-Nordström models. To obtain the stress-energy tensor the 0-components of the Ricci have to be isolated and shifted to the right side of the field equations. The dimensional reduction yields

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[1] For $a_T > 0$. It should be noted that $\mathfrak{p}$ has a pole restricting the parameters to a physically reasonable range.

$$\begin{aligned} R_{mn} = m_m m_n & \left(M_0 B_0 + M_0 C_0 - M_0 E_0 \right) \\ & + b_m b_n \left(B_0 M_0 + B_0 C_0 - B_0 E_0 \right) \\ & + c_m c_n \left(C_0 M_0 + C_0 B_0 - C_0 E_0 \right) \\ & - u_m u_n \left(E_0 M_0 + E_0 B_0 + E_0 C_0 \right) \end{aligned} . \tag{3.5}$$

Therein the unit vectors are

$$m_m = \{1,0,0,0\}, \quad b_m = \{0,1,0,0\}, \quad c_m = \{0,0,1,0\}, \quad u_m = \{0,0,0,0\} \ .$$

Contracting the above relation and exploiting the Einstein tensor one gets with the 0-components of (2.17)

$$T_{mn} = \begin{pmatrix} -p & & & \\ & -p & & \\ & & -p & \\ & & & \mu_0 \end{pmatrix} . \tag{3.6}$$

The hydrostatic pressure

$$\kappa p = -(1 + 2\mathfrak{p}) \frac{1}{\mathcal{R}^2} \tag{3.7}$$

is formally identical with the ones of the Schwarzschild and Reissner-Nordström solutions, but contains via ρ_g the NUT parameter. The energy density

$$\kappa \mu_0 = \frac{3}{\mathcal{R}^2} \tag{3.8}$$

is the same for the three models.

In the last paragraphs the relation of the NUT solution with other solutions was discussed. Afore we address the proper interior NUT solution, we analyze the behavior of the reduced metric on the boundary surface, connecting the interior and exterior metric. The shortest way to do this, is to calculate the quantities a_T and $\mathfrak{p}$ on the boundary surface

$$a_T^g = \cos\eta_g, \quad \mathfrak{p}_g = -\frac{\mathcal{R}_g}{\rho_g} \ . \tag{3.9}$$

The curvature quantities B and C of (2.17) coincide immediately with the ones of the exterior solution if the standard co-ordinate (2.2) is used. Bearing in mind that the 0-co-ordinates have opposite orientations in the two regions and that the angles η and ε have opposite signs we get

$$E_a^g = \left\{ -\frac{1}{\rho_g}, \frac{1}{\rho_g} \tan\varepsilon_g, 0,0,0 \right\} \ , \tag{3.10}$$

an expression known from the exterior solution. If the components of the field equations match on the boundary surface this will be valid also for the metrics and their first derivatives. In general, this is presupposed for matching solutions. At first, one obtains from (3.7)

$$\kappa p_g = -\left(1 - 2\frac{\mathcal{R}_g}{\rho_g}\right)\frac{1}{\mathcal{R}_g^2} \ .$$

Taking into consideration the value (2.11) of ρ_g, the junction condition (2.14), and the definition

$$\tilde{\omega} = \frac{l}{r^2}$$

defined in our previous paper, we finally get the first component of the stress-energy tensor of our reduced exterior solution. Since M_0^{ext} differs from M_0^{int} due to the different curvatures of the exterior and the interior surface this does not apply to the stresses on the surface. Moreover, the energy density has a jump. The first term of

$$\kappa \mu_0^g = \kappa\left(\mu_0^g\right)^{SS} + \tilde{\omega}_g^2 \tag{3.11}$$

corresponds to the Schwarzschild energy density.

4. THE SEED METRIC

In the preceding Sections the basic frame of the interior NUT solution was presented. Both the interior and exterior metrics establish an entire gravitational model endowed with a monopole field. To approach the proper interior solution we start with the reduced metric and we perform an intrinsic transformation. This transformation invokes a rule, how to measure local distances and time intervals. The transformed tetrads have the form

$$\overset{1}{e}_1 = \alpha_D \mathcal{R}, \quad \overset{2}{e}_2 = \alpha_D \mathcal{R}\sin\eta = \alpha_D r = A, \quad \overset{3}{e}_3 = \alpha_D \mathcal{R}\sin\eta\sin\vartheta = A\sin\vartheta, \quad \overset{4}{e}_4 = a_D a_T$$
$$\alpha_D = \frac{A}{r}, \quad a_D = \frac{r}{A} \tag{4.1}$$

and the metric reads as

$$ds^2 = \alpha_D^2\left[\mathcal{R}^2 d\eta^2 + \mathcal{R}^2\sin^2\eta d\vartheta^2 + \mathcal{R}^2\sin^2\eta\sin^2\vartheta d\varphi^2\right] + a_D^2 a_T^2 dit^2 \ . \tag{4.2}$$

In the radial direction of the surface one has

$$dx^1 = \alpha_D \mathcal{R} d\eta = \frac{1}{\cos\eta}\alpha_D dr = \frac{1}{\sqrt{1-\frac{r^2}{\mathcal{R}^2}}}\alpha_D dr, \quad \partial_1 = a_D \cos\eta \frac{\partial}{\partial r} \tag{4.3}$$

and in the direction of the of the local extradimension

$$dx^0 = \alpha_D d\mathcal{R}, \quad \partial_0 = a_D \frac{\partial}{\partial \mathcal{R}} = a_D \sin\eta \frac{\partial}{\partial r} \ . \tag{4.4}$$

For the present, the deformation factor α_D is a function of $\mathcal{R}$ and η

$$\alpha_D = \alpha_D(\mathcal{R}, \eta) \ . \tag{4.5}$$

If one has selected a sphere from the family of spheres and has applied the embedding condition (2.1), α_D will only be a function of η or r, respectively. The deformation factor leads us to a new field strength

$$D_a = \frac{1}{\alpha_D}\alpha_{D|a} = \left\{-\frac{l^2}{A^3}\sin\eta, -\frac{l^2}{A^3}\cos\eta, 0,0,0\right\} \ . \tag{4.6}$$

Moreover, all quantities of the reduced metric have to be worked up and listed

$$\begin{gathered} M_0 = B_0 = C_0 = \frac{a_D}{\mathcal{R}} = \frac{1}{A}\sin\eta \ , \quad E_0 = -\frac{a_D}{\rho_g a_T}\cos\eta \\ B_1 = C_1 = \frac{a_D}{r}\cos\eta = \frac{1}{A}\cos\eta, \quad C_2 = \frac{1}{A}\cot\vartheta, \quad E_1 = -\frac{a_D}{\rho_g a_T}\sin\eta \end{gathered} \ . \tag{4.7}$$

These are the quantities of the reduced metric enhanced by the deformation factor. They have to be rounded out with the quantity (4.6):

$$\begin{gathered} {}^*B_0 = B_0 + D_0 = \frac{a_D^2}{A}\sin\eta, \quad {}^*C_0 = C_0 + D_0, \quad {}^*E_0 = E_0 + D_0 \\ {}^*B_1 = B_1 + D_1 = \frac{a_D^2}{A}\cos\eta, \quad {}^*C_1 = C_1 + D_1, \quad {}^*C_2 = C_2, \quad {}^*E_1 = E_1 + D_1 \end{gathered} \ . \tag{4.8}$$

We remind the reader that all these quantities do not describe a new surface. They are still defined on the surface of the reduced metric endowed with a new rule for measuring. Inserting these quantities into the Ricci one gains new subequations. To facilitate the calculations, we note some formulae[2]. A quantity appears in these relations containing the monopole field strength of the exterior solution

$$\mathcal{H}_{23} = -i a_D \omega \cos\eta, \quad \mathcal{H}^2 = 2\mathcal{H}_{23}\mathcal{H}^{23}, \quad \omega = \frac{l}{A^2} \ . \tag{4.9}$$

Although the seed metric is a static metric it has rotational properties. They are intrinsic properties of the static geometry. The quantity $\mathcal{H}$ differs from the analogous one of the exterior solution by the angular functions and they coincide on the boundary surface of these two solutions. For the subequations of the Ricci we note

$$\begin{aligned}
{}^*B_{1\underset{2}{\|}1} + {}^*B_1{}^*B_1 &= -M_0{}^*B_0 - \mathcal{H}^2 \\
{}^*C_{2\underset{3}{\|}2} + {}^*C_2{}^*C_2 &= -B_0C_0 - D_1D_1 + \mathcal{H}^2 \\
{}^*C^s{}_{\underset{3}{\|}s} + {}^*C^s{}^*C_s &= -M_0{}^*C_0 - B_0C_0 - D_1D_1 \\
{}^*E_{1\underset{4}{\|}1} - {}^*E_1{}^*E_1 &= -M_0{}^*E_0 + 3D_0E_0 + B_1D_1 - 2\mathcal{H}^2 \\
{}^*E^s{}_{\underset{4}{\|}s} - {}^*E^s\,{}^*E_s &= -M_0{}^*E_0 - 2B_0E_0 + D_0E_0 + B_1D_1 - \mathcal{H}^2
\end{aligned} \quad . \tag{4.10}$$

Inserting into the Ricci

$$\begin{aligned}
R_{mn} = &-\left[{}^*B_{n\underset{2}{\|}m} + {}^*B_n{}^*B_m\right] - b_nb_m\left[{}^*B^s{}_{\underset{2}{\|}s} + {}^*B^s{}^*B_s\right] \\
&-\left[{}^*C_{n\underset{3}{\|}m} + {}^*C_n{}^*C_m\right] - c_nc_m\left[{}^*C^s{}_{\underset{3}{\|}s} + {}^*C^s{}^*C_s\right] \\
&+\left[{}^*E_{n\underset{4}{\|}m} - {}^*E_n{}^*E_m\right] + u_nu_m\left[{}^*E^s{}_{\underset{4}{\|}s} - {}^*E^s{}^*E_s\right]
\end{aligned} \tag{4.11}$$

one obtains for the nonvanishing components

$$\begin{aligned}
R_{11} &= \left[M_0{}^*B_0 + M_0{}^*C_0 - M_0{}^*E_0\right] + 3D_0E_0 + B_1D_1 \\
R_{22} &= \left[{}^*B_0M_0 + {}^*B_0C_0 - {}^*B_0E_0\right] + \mathcal{H}^2 + \omega^2 \\
R_{33} &= \left[{}^*C_0M_0 + {}^*C_0C_0 - {}^*C_0E_0\right] + \mathcal{H}^2 + \omega^2 \\
R_{44} &= -\left[{}^*E_0M_0 + {}^*E_0B_0 + {}^*E_0C_0\right] - \mathcal{H}^2 - \omega^2 + E_0D_0 + B_0D_0
\end{aligned} \quad , \tag{4.12}$$

and for the stress-energy tensor

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$$M_0B_0 = \frac{1}{A^2}\sin^2\eta, \quad M_0{}^*B_0 = \frac{a_D^2}{A^2}\sin^2\eta = \frac{a_D^4}{\mathcal{R}^2}, \quad M_0D_0 = -\omega^2\sin^2\eta$$

$$D_1D_1 = \frac{l^2}{A^2}\omega^2\cos^2\eta = \cos^2\eta - a_D^2\cos^2\eta = {}^*B_1D_1 - B_1D_1$$

$$B_1D_1 = -\omega^2\cos^2\eta, \quad {}^*B_1D_1 = -a_D^2\omega^2\cos^2\eta = \mathcal{H}_{23}\mathcal{H}_{23}$$

$$D_{1\|1} = -3\,{}^*B_1D_1 - M_0D_0 = -\mathcal{H}^2 + a_D^2\omega^2\cos^2\eta + \omega^2\sin^2\eta$$

$$\begin{aligned}
\kappa T_{11} &= -2\,{}^*B_0E_0 + B_0C_0 + \mathcal{H}_{23}\mathcal{H}_{23} + \omega^2\cos^2\eta \\
\kappa T_{22} &= -2M_0E_0 + M_0B_0 + 2D_0E_0 - \mathcal{H}_{23}\mathcal{H}_{23} - \omega^2\cos^2\eta \\
\kappa T_{33} &= -2M_0E_0 + M_0C_0 + 2D_0E_0 - \mathcal{H}_{23}\mathcal{H}_{23} - \omega^2\cos^2\eta \\
\kappa T_{44} &= -3M_0{}^*B_0 - \left(\mathcal{H}^2 + \omega^2\right) - \mathcal{H}_{23}\mathcal{H}_{23}
\end{aligned} \quad . \qquad (4.13)$$

The stress-energy tensor has only diagonal components and is covariantly conserved.

5. THE PROPER NUT METRIC

Having defined a spacetime-dependent method of measuring on the surface an additional structure on this surface is implemented by an intrinsic transformation. Doing so only the time-like part of the metric alters

$$dx^4 = a_D a_T\left[2il\left(1-\cos\vartheta d\varphi\right)+idt\right] . \qquad (5.1)$$

It differs from the exterior solution by the gravitational factor a_T. The new tetrads read as

$$\begin{aligned}
&\overset{3}{e}_3 = A\sin\vartheta, \quad \overset{4}{e}_3 = 2il a_D a_T\left(1-\cos\vartheta\right), \quad \overset{4}{e}_4 = a_D a_T \\
&\underset{3}{e}^3 = \frac{1}{A\sin\vartheta}, \quad \underset{3}{e}^4 = -2il\left(1-\cos\vartheta\right), \quad \underset{4}{e}^4 = \alpha_D\alpha_T
\end{aligned} \quad . \qquad (5.2)$$

The Ricci-rotation coefficients have a further contribution

$$H_{mn}{}^s = H_{mn}u^s + H^s{}_m u_n + H^s{}_n u_m \qquad (5.3)$$

with the only component of H

$$H_{23} = -ia_D a_T\omega\ . \qquad (5.4)$$

The field equations are enhanced with[3]

$$R_{34} = H^s{}_{\underset{4}{3\|s}}\ . \qquad (5.5)$$

Since the Maxwell-like equations

$$H^s{}_{\underset{4}{n\|s}} = 0, \quad \mathrm{rot}\vec{H} = 0 \qquad (5.6)$$

are satisfied no energy current exists in the interior of the source. The complete Ricci reads as

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[3] The 4th graded derivative coincides with the space-like derivative.

$$R_{mn} = {}^*R_{mn} + 2u_{(m}H^{s}{}_{n)\underset{4}{\|}s} + \begin{pmatrix} 0 & & & \\ & -H^2 & & \\ & & -H^2 & \\ & & & +H^2 \end{pmatrix}, \quad R = {}^*R - H^2 \qquad (5.7)$$

wherein ${}^*R_{mn}$ and ${}^*R$ are the Ricci and the contracted Ricci of the seed metric. The stress-energy tensor reads as

$$G_{mn} = -\kappa\, {}^*T_{mn} - 2\left[H_m{}^s H_{ns} - \frac{1}{4} g_{mn} H^2\right] + u_m u_n H^2 \ . \qquad (5.8)$$

${}^*T_{mn}$ has the structure (3.6). The new NUT term is fairly Maxwell-like. If one has proved that ${}^*T_{mn}$ is free of divergence the total stress-energy tensor of the NUT model is free of divergence, which is a presupposition. The Maxwell-like NUT term vanishes separately due to the Maxwell-like relations (5.6) and

$$H_{<mn\underset{4}{\|}s>} + {}^*E_{<m}H_{ns>} = 0, \quad \operatorname{div}\vec{H} + {}^*\vec{E}\vec{H} = 0 \ . \qquad (5.9)$$

The NUT field has only one component pointing into the radial direction. One can recognize this by establishing the dual vector

$$H^{\gamma} = \frac{i}{2}\varepsilon_{\alpha\beta}{}^{\gamma}H^{\alpha\beta} = \left\{a_D a_T \omega, 0, 0, 0\right\}, \quad H^1 \stackrel{*}{=} iH_{23}, \quad \alpha = 1,2,3 \ . \qquad (5.10)$$

Lastly, the values of the basic quantities are calculated on the boundary surface. The intricate quantity

$$a_T^g = \left[\left(\mathcal{R}_g + \rho_g\right)\cos\eta_g - \mathcal{R}_g \cos\eta_g\right]\frac{1}{\rho_g} = \cos\eta_g = \cos\varepsilon_g \qquad (5.11)$$

becomes considerably simpler on the boundary surface. Therefore, the 0-components of the field strengths read as

$$M_0^g = B_0^g = C_0^g = \frac{1}{A_g}\sin\eta_g = -\frac{1}{A_g}\sin\varepsilon_g, \quad E_0^g = \frac{a_D}{\rho_g} \ . \qquad (5.12)$$

Firstly, one has to bear in mind that M_0^g does not match the corresponding quantity of the exterior geometry because the exterior and interior surfaces are rather different and have different radial curvatures. Secondly, one has to take into consideration that the local 0-directions of both co-ordinate systems have opposite orientations and the corresponding 0-components of the two solutions have opposite signs. The 1-components read as

$$B_1^g = C_1^g = \frac{1}{A_g}\cos\eta_g = \frac{1}{A_g}\cos\varepsilon_g, \quad E_1^g = -\frac{a_D}{\rho_g}\tan\eta_g = \frac{a_D}{\rho_g}\tan\varepsilon_g \ . \qquad (5.13)$$

Taking advantage of the fact that the NUT field strengths of both solutions have the same value

$$H^g_{23} = \mathcal{H}^g_{23} \tag{5.14}$$

on the boundary surface, so that they go off continuously from the exterior to the interior region, one gets with the help of (4.13), (5.7), and (5.12) after some algebra

$$\kappa T^{\ g}_{mn} = \begin{pmatrix} 0 & & & \\ & -2a^2_{Dg}\tilde{\omega}^2_g & & \\ & & -2a^2_{Dg}\tilde{\omega}^2_g & \\ & & & a^4_{Dg}\left(\frac{3}{\mathcal{R}^2_g} + \tilde{\omega}^2_g\right) \end{pmatrix} . \tag{5.15}$$

The radial hydrostatic pressure vanishes on the boundary surface. This provides the stability of the object. Putting zero the NUT parameter one obtains the Schwarzschild values of the stress-energy tensor on the boundary surface.

It is of some interest to recast the stress-energy tensor to be comparable with the stress-energy tensor of the Schwarzschild model. If one takes the first component from the relation (4.13) of the seed metric and supplements it with (5.8) in order to get the proper NUT metric one attains

$$\kappa T_{11} = -2\,{}^*B_0E_0 + B_0C_0 + \mathcal{H}_{23}\mathcal{H}_{23} - H_{23}H_{23} + \omega^2\cos^2\eta \ .$$

For the reduced metric one reads from (3.4) $E_0 = -\mathcal{P}/\mathcal{R}$. One recognizes with (4.7) that for the proper metric one has to write

$$E_0 = -a_D\frac{\mathcal{P}}{\mathcal{R}} \ . \tag{5.16}$$

For the first term of the above equation one obtains with (4.7)

$$-2\,{}^*B_0E_0 = -2B_0E_0 - 2D_0E_0 = 2\left[a^2_D\frac{\mathcal{P}}{\mathcal{R}^2} - i\frac{l^2}{A^3}\sin\eta\, a_D\frac{\mathcal{P}}{\mathcal{R}}\right] = 2\mathcal{P}\frac{a^2_D}{\mathcal{R}^2} - 2\mathcal{P}\omega^2\sin^2\eta$$

and lastly,

$$\kappa T_{11} = -\kappa p^{SS} - (1+2\mathcal{P})\,\omega^2\sin^2\eta + \omega^2 + \mathcal{H}_{23}\mathcal{H}_{23} - H_{23}H_{23} \ . \tag{5.17}$$

Therein

$$\kappa p^{SS} = -(1+2\mathcal{P})\frac{a^2_D}{\mathcal{R}^2} \tag{5.18}$$

is the Schwarzschild contribution of the hydrostatic pressure. For the other components of the stress-energy tensor one obtains in the same manner

$$\begin{aligned} \kappa T_{22} = \kappa T_{33} &= -\kappa p^{SS} + (1+2\mathcal{P})\,\omega^2\sin^2\eta - \omega^2 - \mathcal{H}_{23}\mathcal{H}_{23} + H_{23}H_{23} \\ \kappa T_{44} &= \kappa\mu_0 + \omega^2 + 3\mathcal{H}_{23}\mathcal{H}_{23} - 3H_{23}H_{23} \end{aligned} \ . \tag{5.19}$$

Therein

$$\kappa\mu_0 = a_D^4 \frac{3}{\mathcal{R}^2} \tag{5.20}$$

is the Schwarzschild-like energy density. The last conversion makes clear the relations with, and differences from the Schwarzschild model. Only a few authors have dealt with the interior solution of the NUT metric. We have found a paper of Lukacs, Newman, Spaling, and Winicour [4]. They have described an interior model with a rigid rotating fluid.

6. CONCLUSIONS

We derived an interior solution for the NUT metric by geometrical means. Once known the geometrical structure of the exterior solutions, it is easy to derive the interior solution by exploiting the curvature properties of the exterior solution on the boundary surface. Since we have developed a general technique for constructing interiors for known solutions, which are embeddable in higher dimensional space, it is sufficient to use these results and adapt them to the NUT model.

7. REFERENCES

1. Burghardt R., *The geometrical interpretation of the NUT metric*.
   http://arg.or.at Report ARG-2009-02
2. Burghardt R., *New embedding of Schwarzschild geometry. II. Interior solution.*
   http://arg.or.at Report ARG-2001-03 and Science Edition, Potsdam 2006, p 29
3. Burghardt R., *Reissner exterior and interior*.
   http://arg.or.at Report ARG-2007-02 and Tensor **71**, 216, 2009
4. Lukacs B. , Newman E. T., Spaling G., and Winicour J*., A NUT-like solution with fluid matter*. Hung. Ac. Sci. Kfki-62, 1982