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REMARKS ON THE MODEL OF OPPENHEIMER AND SNYDER I

Rainer Burghardt*

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Contents

We will reinvestigate the collapsing model of Oppenheimer and Snyder. We will show that the collapsing stellar object is infinitely large at the start of the collapse. It collapses in free fall and its surface reaches the event horizon after infinitely long proper time.

* e-mail: arg@aon.at, home page: http://arg.or.at/

1. INTRODUCTION

In 1939, Oppenheimer and Snyder [1] presented a paper now known as that paper which has given rise to the theory of black holes, although the term 'black hole' has been introduced much later. Moreover, the OS approach differs from current methods to implement black holes. The OS model is composed of a collapsing interior and an exterior solution, where the exterior part is the Schwarzschild solution, which is due to the Birkhoff theorem also valid even if the field generating stellar object collapses. Most approaches to a black hole do not use an interior solution. In this case the exterior Schwarzschild solution or the exterior Kerr solution is extended beneath the event horizon. The inner region of this solution is to describe a black hole.

The OS model is based on an existing solution by Tolman [2]. The stellar object is made of pressure-free dust with homogeneous density. Since in this case, the internal resistance against a contraction is missing, the object cannot be static. It collapses as a consequence of its own gravitational attraction.

A completely pressure-free star is not physically realistic, as pressure can be expected at a sufficiently high density. A pressure-free stellar object may approximately describe a dying star. If the thermonuclear processes are exhausted inside a star, they give way to the star’s own gravitational attraction, and the star collapses. The just-discussed simplification to $p = 0$ is primarily on practical grounds. The integration of Einstein's field equations without this condition leads to considerable difficulties, and it is hard to find an analytical solution.

In addition to these above-mentioned limitations the OS model permits further criticism. The star collapses with the velocity of observers coming in free fall from infinity. The stellar object at the time $t = 0$ would have been infinitely large. We will work out this in the following. Mitra [3] has shown that inconsistencies in the OS model can only be resolved if the mass of a hypothetical OS black hole is $M = 0$. However, a massless star is contrary to the widespread opinion that black holes are supermassive objects. For these and other reasons which we will point out the OS model does not provide an appropriate basis for a black hole.

2. THE OS INTERIOR SOLUTION, BASIC RELATIONS

The first considerations are closely related to the original paper by OS, but we have to introduce auxiliary variables which are in close connection with geometrical quantities. Both, the interior and the exterior solutions are treated in two different co-ordinate systems: the one is comoving with the collapsing matter and the other one is not comoving. It is the very transition between the two systems which brings insight into how the collapse proceeds, and thus, sheds light onto the inconsistencies of the model.

For the comoving and for the non-comoving co-ordinate system we use the notations

$$\{r', \vartheta, \varphi, t'\}, \quad \{r, \vartheta, \varphi, t\} \ . \tag{2.1}$$

In the comoving system we write the line element according to OS as

$$ds^2 = e^{\bar{\omega}} dr'^2 + e^{\omega}\left(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right) - dt'^2, \quad \bar{\omega} = \bar{\omega}(r', t'), \quad \omega = \omega(r', t') \; . \tag{2.2}$$

For the two metric factors OS put

$$e^{\omega} = \left(G + Ft'\right)^{4/3}, \quad e^{\bar{\omega}} = \frac{1}{4} e^{\omega}\left(\frac{\partial \omega}{\partial r'}\right)^2 \; , \tag{2.3}$$

wherein one has for the interior solution

$$G = \sqrt{r'^3}, \quad F = -\frac{3}{2}\sqrt{2M}\sqrt{\frac{r'^3}{r'^3_g}} \tag{2.4}$$

with r'_g the value of r' on the surface of the stellar object, i.e. at the boundary of the interior and exterior solutions. We note the auxiliary variables

$$\mathcal{R}'_g = \sqrt{\frac{r'^3_g}{2M}}, \quad \rho'_g = 2\mathcal{R}'_g, \quad \Lambda = 1 - \frac{3}{\rho'_g} t' \; . \tag{2.5}$$

As in the non-comoving system the lateral part of the metric has the form

$$r^2\left(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right) \; ,$$

and this form is conserved under a co-ordinate tranformation between these two systems. A comparison with (2.2) gives

$$r^2 = e^{\omega} \; . \tag{2.6}$$

With (2.3) and (2.5) we obtain the relation

$$r = \Lambda^{2/3} r' \; . \tag{2.7}$$

From the point of view of the co-moving observer the radial co-ordinate of the surface does not change

$$\frac{\partial r'_g}{\partial r'} = 0, \quad \frac{\partial r'_g}{\partial t'} = 0 \; . \tag{2.8}$$

For the quantity Λ one gets the relations

$$\frac{\partial \Lambda}{\partial r'} = 0, \quad \frac{\partial \Lambda}{\partial t'} = -\frac{3}{\rho'_g} \; , \tag{2.9}$$

which we need for some calculations. For the radial co-ordinate of the non-comoving observer system one gains, taking advantage of (2.7) and the above formulae,

$$\frac{\partial r_g}{\partial r'} = \frac{\partial}{\partial r'}\left(\Lambda^{2/3} r'_g\right) = 0, \quad \frac{\partial r_g}{\partial t'} = \frac{\partial}{\partial t'}\left(\Lambda^{2/3} r'_g\right) = -\frac{r_g}{\mathcal{R}_g} = -\sqrt{\frac{2M}{r_g}} \; . \tag{2.10}$$

In this calculation the relations

$$\mathcal{R}_g = \Lambda \mathcal{R}'_g, \quad \rho_g = \Lambda \rho'_g \tag{2.11}$$

have been used which can be verified with (2.5) and (2.7). Thus, we have also motivated the introduction of these auxiliary variables.

$$\rho = \sqrt{\frac{2r^3}{M}} \qquad (2.12)$$

is the curvature radius of the Schwarzschild parabola. $\mathcal{R}_g$ has half the length of ρ_g, the curvature vector at the boundary. If one extends the curvature vector of the Schwarzschild parabola to the directrix of the Schwarzschild parabola, the resulting distance between the Schwarzschild parabola and the directrix has the length

$$\mathcal{R} = \sqrt{\frac{r^3}{2M}} \ . \qquad (2.13)$$

(2.11) refers to the values on the boundary surface. The quantities occurring in the OS model allow a geometric interpretation which facilitates the understanding of the theory. Since the proper time T' coincides with the co-ordinate time t' in the comoving system, the second relation (2.10) can be written as

$$\frac{\partial r_g}{\partial t'} = v_g, \quad v_g = -\sqrt{\frac{2M}{r_g}} \qquad (2.14)$$

whereby v_g is the velocity of an observer who is in free fall in the Schwarzschild field, coming from the infinite and reaching the surface of the stellar object. However, this means that the surface itself has this speed and must come from infinity. After a brief calculation we found out an inconsistency of the OS and model. This problem will be represented in detail later on.

Using (2.10) the changes of further variables which relate to the surface can be calculated. We summarize the results with

$$\frac{\partial r_g}{\partial r'} = 0, \quad \frac{\partial r_g}{\partial t'} = v_g, \quad \frac{\partial \mathcal{R}_g}{\partial r'} = 0, \quad \frac{\partial \mathcal{R}_g}{\partial t'} = -\frac{3}{2}, \quad \frac{\partial \rho_g}{\partial r'} = 0, \quad \frac{\partial \rho_g}{\partial t'} = -3 \ . \qquad (2.15)$$

In the next step we calculate the changes of the relevant variables in the interior of the stellar object. From (2.7) and (2.9) one gets

$$\frac{\partial r}{\partial r'} = \Lambda^{2/3} = \frac{r}{r'}, \quad \frac{\partial r}{\partial t'} = -\frac{r}{\mathcal{R}_g} = v_I \ . \qquad (2.16)$$

v_I is the speed in the interior, i.e. the speed with which the particles draw near in the interior during the collapse. The velocity decreases linearly inwards, and in the center of the object one has

$$v_I(0) = 0 \ . \qquad (2.17)$$

Finally, the second relation (2.3) should be resolved. According to (2.16) one has

$$\frac{\partial e^{\omega/2}}{\partial r'} = \frac{r}{r'}$$

and at last

$$e^{\varpi} = \Lambda^{4/3} = \frac{r^2}{r'^2} \ . \qquad (2.18)$$

Hence all the metric coefficients of the inner OS solution can be written in a form that simplifies the further considerations

$$\overset{1'}{e}_{1'}=\frac{r}{r'},\quad \overset{2'}{e}_{2'}=r,\quad \overset{3'}{e}_{3'}=r\sin\vartheta,\quad \overset{4'}{e}_{4'}=1\ . \tag{2.19}$$

The metric can be written either as

$$ds^2=\frac{r^2}{r'^2}dr'^2+r^2d\vartheta^2+r^2\sin^2\vartheta d\varphi^2-dt'^2 \tag{2.20}$$

or as

$$ds^2=\Lambda^{4/3}\left[dr'^2+r'^2d\vartheta^2+r'^2\sin^2\vartheta d\varphi^2\right]-dt'^2\ . \tag{2.21}$$

The latter form is entirely written in comoving co-ordinates and the factor Λ contains the time dependence of the otherwise ‘flat’ 3-dimensional line element.

3. THE OS INTERIOR SOLUTION, CO-ORDINATE AND REFERENCE SYSTEMS

In the previous Section we have already made use of the variable r which designates the radial co-ordinate in the non-comoving system. OS have specified the relation of t' and t. Thus, a matrix can be assembled for the transformation between the two co-ordinate systems. The finding of such a transformation is obviously quite tedious. It is probably for this reason that other authors did not provide such a co-ordinate transformation for their models, or the putting up of a co-ordinate transformation was not possible because the model has no analytical solution. The use of different co-ordinate systems is apparently the only purpose of providing calculations on the simplest possible basis. The great advantage lies in the fact that such a co-ordinate transformation is accompanied by a Lorentz transformation which contains the velocity parameters. If one has found such a Lorentz transformation, and if one refers to the velocity of the surface of the stellar object, one has the *physical* velocity of the collapse at hand.

In the last Section we have already prepared the way for a Lorentz transformation. From (2.16) and (2.19) we obtain

$$dr=\frac{r}{r'}dr'-\frac{r}{\mathcal{R}_g}dt'=dx^{1'}-iv_I dx^{4'},\quad v_I=-\frac{r}{\mathcal{R}_g}\ , \tag{3.1}$$

wherein the $\{dx^{1'},dx^{4'}\}$ are the anholonomic tetrad differentials. By means of an auxiliary variable

$$y=\frac{1}{2}\left[\left(\frac{r'}{r'_g}\right)^2-1\right]+\frac{r'_g}{2M}\frac{r}{r'},\quad dy=\frac{r'}{r'^2_g}dr'-\frac{1}{2M}\sqrt{\frac{2M}{r_g}}dt' \tag{3.2}$$

and from

$$t=\frac{3}{2}\frac{1}{\sqrt{2M}}\left[r'^{3/2}_{g}-(2M)^{3/2}\right]-4M\sqrt{y}+2M\ln\frac{\sqrt{y}+1}{\sqrt{y}-1} \tag{3.3}$$

one gets, by differentiating,

$$dt=-2M\frac{y^{3/2}}{y-1}\frac{r'}{r'_g}dr'+\sqrt{\frac{2M}{r_g}}\frac{y^{3/2}}{y-1}dt' \;. \tag{3.4}$$

From this one can read the transformation coefficients of the co-ordinate transformation

$$\Lambda^{i}_{i'}=x^{i}_{|i'} \;. \tag{3.5}$$

Since the coefficients are orthogonal, one gains also the reciprocal values

$$\Lambda^{1}_{1'}=\frac{r}{r'},\quad \Lambda^{1}_{4'}=i\frac{r}{\mathcal{R}_g}=-iv_I,\quad \Lambda^{4}_{1'}=-2Mi\frac{y^{3/2}}{y-1}\frac{r'}{r'^2_g},\quad \Lambda^{4}_{4'}=\frac{y^{3/2}}{y-1}\sqrt{\frac{2M}{r_g}}$$

$$\Lambda^{1'}_{1}=\alpha_I^2\frac{r'}{r},\quad \Lambda^{4'}_{1}=-i\alpha_I^2 v_I,\quad \Lambda^{1'}_{4}=-i\alpha_I^2\frac{y-1}{y^{3/2}}\frac{r'}{r_g},\quad \Lambda^{4'}_{4}=\alpha_I^2\frac{y-1}{y^{3/2}}\sqrt{\frac{r_g}{2M}} \;. \tag{3.6}$$

$$\alpha_I=\frac{1}{\sqrt{1-\frac{2M}{r}\frac{r'^3}{r'^3_g}}}=\frac{1}{\sqrt{1-\frac{r^2}{\mathcal{R}_g^2}}},\quad v_I=-\frac{r}{\mathcal{R}_g}$$

In these expressions all indices are co-ordinate indices. From

$$g^{ik}=\Lambda^{i}_{i'}{}^{k}_{k'}g^{i'k'}$$

and (2.19) can be calculated the 4-beine $\underset{m}{e}^{i}$ and from these the reciprocal ones

$$\overset{1}{e}_1=\alpha_I,\quad \overset{2}{e}_2=r,\quad \overset{3}{e}_3=r\sin\vartheta,\quad \overset{4}{e}_4=\alpha_I\sqrt{\frac{r_g}{2M}}\frac{y-1}{y^{3/2}} \;. \tag{3.7}$$

The indices m number the 4-beine of the reference system which is associated with the static observers. Now it is easy to calculate the corresponding Lorentz transformation connecting the comoving observers and the observers at rest

$$L^{m}_{m'}=\overset{m}{e}_i\,\Lambda^{i}_{i'}\,\underset{m'}{e}^{i'}$$
$$L^{1}_{1'}=\alpha_I,\quad L^{1}_{4'}=-i\alpha_I v_I,\quad L^{4}_{1'}=i\alpha_I v_I,\quad L^{4}_{4'}=\alpha_I \;. \tag{3.8}$$

Thus, one has the physical quantities describing the collapse at hand. On the surface one has with

$$\overset{1}{e}_1=\alpha_I^g=\frac{1}{\sqrt{1-\frac{r_g^2}{\mathcal{R}_g^2}}}=\frac{1}{\sqrt{1-\frac{2M}{r_g}}},\quad \overset{4}{e}_4=\alpha_I^g\sqrt{\frac{r_g}{2M}}\frac{\frac{r_g}{2M}-1}{\sqrt{\left(\frac{r_g}{2M}\right)^2}}=\sqrt{1-\frac{2M}{r_g}}$$

the Schwarzschild values of the exterior field on the surface of the stellar object. Thus, the two solutions are matched on the boundary surface.

For the velocity on the surface one obtains

$$v_l^g = -\frac{r_g}{\mathcal{R}_g} = -\sqrt{\frac{2M}{r_g}} \quad , \tag{3.9}$$

the velocity of an observer who is in free fall from infinity. Thus, we have once again proved by using physically relevant equations that the surface of the stellar object collapses in free fall from infinity. From (3.9) we also recognize that for $r_g = \infty$ the initial velocity is $v_l^g = 0$. The OS stellar object in its initial state would have an infinitely large extension, collapses in free fall and leaves empty space behind it in which a Schwarzschild field spreads. However, the collapse velocity would reach the speed of light at $r_g = 2M$ which has to be ruled out by the principle of relativity. At this location the gravity and tidal forces would be infinitely large. Under these conditions a star cannot exist. The OS model is afflicted with all those problems which are known from the Schwarzschild theory. Consequently, the OS model cannot be used as a base model for a black hole.

4. THE OS EXTERIOR SOLUTION

For the exterior solution OS start from the same metric (2.2) with the ansatz (2.3) where now

$$G = \sqrt{r'^3}, \quad F = -\frac{3}{2}\sqrt{2M}, \quad \Lambda = 1 - \frac{3}{\rho'}t', \quad \rho' = \sqrt{\frac{2r'^3}{M}} \tag{4.1}$$

and the following auxiliary formulae

$$\frac{\partial \Lambda}{\partial r'} = \frac{9}{4}\sqrt{\frac{2M}{r'^5}}t', \quad \frac{\partial \Lambda}{\partial t'} = -\frac{3}{\rho'}, \quad \frac{\partial \omega}{\partial r'} = \frac{2}{\Lambda r'} \tag{4.2}$$

apply. After similar calculations such as we have performed for the interior solution we obtain the tetrads in the comoving system

$$\overset{1'}{e}_{1'} = \sqrt{\frac{r'}{r}} = \Lambda^{-1/3}, \quad \overset{2'}{e}_{2'} = r, \quad \overset{3'}{e}_{3'} = r\sin\vartheta, \quad \overset{4'}{e}_{4'} = 1 \tag{4.3}$$

and from

$$r = \Lambda^{2/3} r' \tag{4.4}$$

the auxiliary formulae

$$\frac{\partial r}{\partial r'} = \sqrt{\frac{r'}{r}}, \quad \frac{\partial r}{\partial t'} = -\frac{r}{\mathcal{R}} = -\sqrt{\frac{2M}{r}} = v_E \quad , \tag{4.5}$$

in which we recognize the Schwarzschild velocity of free fall from infinity. OS put for the time of the static system

$$t = \frac{3}{2}\frac{1}{\sqrt{2M}}\left(r'^{3/2} - r^{3/2}\right) - 2\sqrt{2Mr} - 2M\ln\frac{\sqrt{r} - \sqrt{2M}}{\sqrt{r} + \sqrt{2M}} \quad . \tag{4.6}$$

With (4.4) and (4.1), last equation, one has

$$r=\left(1-3\sqrt{\frac{M}{2r'^3}}t'\right)^{2/3}r'\ .$$

Isoliating t'

$$t'=\frac{2}{3}\frac{1}{\sqrt{2M}}\left(r'^{3/2}-r^{3/2}\right) \tag{4.7}$$

we finally obtain

$$t'=t+2\sqrt{2Mr}+2M\ln\frac{1-\sqrt{\frac{2M}{r}}}{1+\sqrt{\frac{2M}{r}}}\ , \tag{4.8}$$

the formula for the transition from the static Schwarzschild time co-ordinate to a time co-ordinate which refers to an observer who is in free fall from infinity. The relation originally was found by Lemaître. By a gauge transformation of the radial co-ordinate

$$r''=\frac{2}{3}\frac{1}{\sqrt{2M}}r'^{3/2},\quad t''=t' \tag{4.9}$$

one obtains with

$$\overset{1'}{e}_{1'}\,dr'=\sqrt{\frac{r'}{r}}\sqrt{\frac{2M}{r'}}dr''=\sqrt{\frac{2M}{r}}dr''$$

the relation

$$dx^{1''}=-v_E dr'',\quad v_E=-\sqrt{\frac{2M}{r}}\ , \tag{4.10}$$

well-known from the Lemaître transformation as well.

After recasting one can specify r as a function of the Lemaître co-ordinates

$$r=\sqrt[3]{2M}\left[\frac{3}{2}\left(r''-t''\right)\right]^{2/3} \tag{4.11}$$

and can write the curvature radius of the Schwarzschild parabola as

$$\rho=3\left(r''-t''\right)\ . \tag{4.12}$$

Thus, we have explained the OS co-ordinate transformation as a Lemaître transformation. With the help of (4.5) and by differentiating (4.8) we evaluate the coefficients of this transformation. By transvecting with the 4-bein of the comoving system

$$\overset{1''}{e}_{1''}=-v_E,\quad \overset{4''}{e}_{4''}=1 \tag{4.13}$$

and the 4-bein of the static Schwarzschild system

$$\overset{1}{e}_{1}=\alpha_E,\quad \overset{4}{e}_{4}=\frac{1}{\alpha_E},\quad \alpha_E=\frac{1}{\sqrt{1-\frac{2M}{r}}} \tag{4.14}$$

we obtain the coefficients of a Lorentz transformation

$$L^{1}_{1''} = \alpha_E, \quad L^{1}_{4''} = -i\alpha_E v_E, \quad L^{4}_{1''} = i\alpha_E v_E, \quad L^{4}_{4''} = \alpha_E \ . \tag{4.15}$$

In these coefficients are included the *physical* components of the relative velocity v_E of the two systems which we refer again to the surface of the stellar object. One has

$$v^{g}_{E} = -\sqrt{\frac{2M}{r_g}} \ . \tag{4.16}$$

In accordance with previous results the surface has always the speed of an object coming in free fall from infinity. At $r_g = 2M$ arise the well-known Schwarzschild problems.

5. SUMMARY

The solution of Oppenheimer and Snyder is an analytical solution of the Einstein field equations which describes alike the interior and the exterior of a stellar object. It has the deficiency of being restricted to pressure-free matter. It is physically unrealistic since the stellar object is infinitely large and its matter is infinitely thinned at the time $t' = 0$. The collapse takes place in free fall. The surface of the object can only asymptotically approach the event horizon, the formation of a black hole is not possible.

6. REFERENCES

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