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REMARKS ON THE MODEL OF OPPENHEIMER AND SNYDER II

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Contents

We calculate the field strengths of the Oppenheimer-Snyder model with the help of tetrads and Ricci-rotation coefficients. We set up the field equations in a covariant manner and we perform a [3+1] decomposition of the Einstein field equations.

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1. INTRODUCTION

In a previous paper [1] we have critically examined the model of Oppenheimer and Snyder [2]. We have worked out that the model describes a collapsing stellar object that initially has infinite extent. When it collapses it leaves behind a Schwarzschild field and its surface reaches the speed of light at the event horizon of the Schwarzschild geometry. At this location the gravity would be infinitely large. Due to these reasons the model is not physically realistic. However, we will concisely calculate the field quantities and the field equations. Despite the shortcomings inherent in the OS model, its detailed description can be constructive. One can study mechanisms which are useful for other models.

2. PRELIMINARIES

OS have found a solution for the interior of a collapsing star. They have shown that a matching of the Schwarzschild exterior at the surface of the star is possible. They have treated the problem in comoving and non-comoving co-ordinates as well. We start with comoving co-ordinates and we write down the metric for the interior using the results of OS.

From the metric

$$ds^2 = \frac{r^2}{r'^2} dr'^2 + r^2 d\vartheta^2 + r^2 \sin^2\vartheta d\varphi^2 - dt'^2 \qquad (2.1)$$

we read the 4-bein

$$\overset{1'}{e}_{1'} = \frac{r}{r'}, \quad \overset{2'}{e}_{2'} = r, \quad \overset{3'}{e}_{3'} = r\sin\vartheta, \quad \overset{4'}{e}_{4'} = 1 \ . \qquad (2.2)$$

For the calculation of the field strengths it is easier to use the non-comoving radial co-ordinate r instead of the comoving co-ordinate r'. The relations

$$r = \Lambda^{2/3} r', \quad \Lambda = 1 - \frac{3}{\rho'_g} t', \quad \rho'_g = 2\mathcal{R}'_g, \quad \mathcal{R}'_g = \sqrt{\frac{r'^3_g}{2M}} \ . \qquad (2.3)$$

are valid. The co-ordinate r is dependent on the time t', while r'_g is the value of the comoving co-ordinate r' on the surface of the collapsing stellar object and is a constant. At the beginning of the collapse we have

$$\Lambda(0) = 1, \quad r_g = r'_g \ .$$

With the quantities

$$\mathcal{R}_g = \Lambda \mathcal{R}'_g = \sqrt{\frac{r_g^3}{2M}}, \quad \rho_g = \Lambda \rho'_g = \sqrt{\frac{2r_g^3}{M}} \ , \qquad (2.4)$$

explained in paper [1], we are able to set up some formulae in a compact way of writing. For further calculations we need several auxiliary formulae. With the help of the Lorentz transformation

$$L^{1}_{1'} = \alpha_I, \quad L^{1}_{4'} = -i\alpha_I v_I, \quad L^{4}_{1'} = i\alpha_I v_I, \quad L^{4}_{4'} = \alpha_I \tag{2.5}$$

with the velocity of the particles in the interior of the stellar object and the associated Lorentz factor

$$v_I = -\frac{r}{\mathcal{R}_g}, \qquad \alpha_I = \frac{1}{\sqrt{1-\frac{r^2}{\mathcal{R}_g^2}}} \tag{2.6}$$

one calculates for the non-comoving system

$$r_{|m} = L^{m'}_{m} e^{i'}_{m'} r_{|i'} \tag{2.7}$$

the change of the static radial co-ordinate

$$r_{|m} = \{1/\alpha, 0, 0, 0\} . \tag{2.8}$$

The marker I will be omitted at the quantities v and α, because we are dealing exclusively with the interior-OS solution in this Section. Similarly, one can determine the values on the surface, if one uses the results of paper [1]

$$\frac{\partial r_g}{\partial r'} = 0, \quad \frac{\partial r_g}{\partial t'} = v_g, \quad \frac{\partial \mathcal{R}_g}{\partial r'} = 0, \quad \frac{\partial \mathcal{R}_g}{\partial t'} = -\frac{3}{2}, \quad \frac{\partial \rho_g}{\partial r'} = 0, \quad \frac{\partial \rho_g}{\partial t'} = -3 . \tag{2.9}$$

One gets

$$r_{g|m} = \{-\alpha v v_g, 0, 0, -i\alpha v_g\} . \tag{2.10}$$

The 4-velocity inside the object relative to the static system is

$$'u_m = \{-i\alpha v, 0, 0, \alpha\} . \tag{2.11}$$

Thus, one has

$$\mathcal{R}_{g|m} = \frac{3}{2} i\, 'u_m, \quad \rho_{g|m} = 3i\, 'u_m . \tag{2.12}$$

With (2.6) we get

$$v_{|m'} = \left\{-\frac{1}{\mathcal{R}_g}, 0, 0, iv\frac{1}{\rho_g}\right\} = \left\{\frac{v_g}{r_g}, 0, 0, iv\frac{1}{\rho_g}\right\} . \tag{2.13}$$

The change of v consists of a circular and a parabolic part. The spatial change of v takes place when one proceeds on a radial line in the interior of the object at a particular time. However, the change in time takes place, because the boundary of the sphere slides down the Schwarzschild parabola with the curvature radius ρ_g. In the static system one has

$$v_{|m} = \left\{-\frac{1}{\alpha\mathcal{R}_g}, 0, 0, 0\right\} - 3iv\frac{1}{\rho_g}\, 'u_m . \tag{2.14}$$

In addition, this gives the values for the change of the Lorentz factor

$$\alpha_{|m'} = \left\{-\alpha^3 v \frac{1}{\mathcal{R}_g}, 0, 0, -i\alpha^3 v^2 \frac{1}{\rho_g}\right\}, \quad \alpha_{|m} = \left\{-\alpha^4 v \frac{1}{\mathcal{R}_g} - \alpha^4 v^3 \frac{1}{\rho_g}, 0, 0, -3i\alpha^4 v^2 \frac{1}{\rho_g}\right\} . \qquad (2.15)$$

Thus, we are ready to set up the field equations in both systems. For the comoving system, which we will treat below, a fairly simple structure of the field equations results.

3. FIELD EQUATIONS, COMOVING SYSTEM

With the help of the tetrads (2.2) we calculate the Ricci-rotation coefficients

$$U_{4'} = A_{1'4'}{}^{1'} = -\overset{1'}{e}_{1'} e^{1'}_{1'|4'}, \quad B_{m'} = A_{2'm'}{}^{2'} = -\overset{2'}{e}_{2'} e^{2'}_{2'|m'} \quad C_{m'} = A_{3'm'}{}^{3'} = -\overset{3'}{e}_{3'} e^{3'}_{3'|m'} . \qquad (3.1)$$

This results in

$$U_{m'} = \left\{0, 0, 0, \frac{i}{\mathcal{R}_g}\right\}, \quad B_{m'} = \left\{\frac{1}{r}, 0, 0, -\frac{iv}{r}\right\}, \quad C_{m'} = \left\{\frac{1}{r}, \frac{1}{r}\cot\vartheta, 0, -\frac{iv}{r}\right\} . \qquad (3.2)$$

Inserting $v = -r/\mathcal{R}_g$ one has

$$U_{4'} = B_{4'} = C_{4'} = \frac{i}{\mathcal{R}_g} \qquad (3.3)$$

With the unit vectors

$$'m_{m'} = \{1, 0, 0, 0\}, \quad 'b_{m'} = \{0, 1, 0, 0\}, \quad 'c_{m'} = \{0, 0, 1, 0\} \qquad (3.4)$$

and the Ricci-rotation coefficients

$$\begin{aligned} A_{m'n'}{}^{s'} &= U_{m'n'}{}^{s'} + B_{m'n'}{}^{s'} + C_{m'n'}{}^{s'} \\ U_{m'n'}{}^{s'} &= {}'m_{m'} U_{n'} {}'m^{s'} - {}'m_{m'} {}'m_{n'} U^{s'} \\ B_{m'n'}{}^{s'} &= {}'b_{m'} B_{n'} {}'b^{s'} - {}'b_{m'} {}'b_{n'} B^{s'} \\ C_{m'n'}{}^{s'} &= {}'c_{m'} C_{n'} {}'c^{s'} - {}'c_{m'} {}'c_{n'} U^{s'} \end{aligned} \qquad (3.5)$$

we form the graded derivatives

$$U_{m'\underset{1}{||}n'} = U_{m'|n'}, \quad B_{m'\underset{2}{||}n'} = B_{m'|n'} - U_{n'm'}{}^{s'} B_{s'}, \quad C_{m'\underset{3}{||}n'} = C_{m'|n'} - U_{n'm'}{}^{s'} C_{s'} - B_{n'm'}{}^{s'} C_{s'} . \qquad (3.6)$$

With the submatrices of the metric

$$h_{m'n'} = \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 1 \end{pmatrix}, \quad {}^3g_{m'n'} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 0 \end{pmatrix} \qquad (3.7)$$

one finally obtains the Ricci

$$
\begin{aligned}
R_{m'n'} = & -h_{m'n'}\left[U^{s'}{}_{\underset{1}{||}s'} + U^{s'}U_{s'}\right] \\
& -\left[B_{n'\underset{2}{||}m'} + B_{n'}B_{m'}\right] - {}'b_{n'}\,{}'b_{m'}\left[B^{s'}{}_{\underset{2}{||}s'} + B^{s'}B_{s'}\right] \\
& -\left[C_{n'\underset{3}{||}m'} + C_{n'}C_{m'}\right] - {}'c_{n'}\,{}'c_{m'}\left[C^{s'}{}_{\underset{3}{||}s'} + C^{s'}C_{s'}\right] \\
= & \frac{3}{2}\frac{1}{\mathcal{R}_g^2}\left[{}^3g_{m'n'} - {}'u_{m'}\,{}'u_{n'}\right]
\end{aligned}
\tag{3.8}
$$

and

$$
R = \frac{3}{\mathcal{R}_g^2} \ . \tag{3.9}
$$

For the Einstein tensor one obtains

$$
G_{\alpha'\beta'} = 0, \quad G_{\alpha'4'} = 0, \quad G_{4'4'} = -\kappa\mu_0, \quad \alpha' = 1', 2', 3' \ . \tag{3.10}
$$

The pressure-free model contains only the time-dependent energy density

$$
T_{44} = \mu_0(t') \ .
$$

OS put

$$
\kappa\mu_0 = \frac{4}{3}\frac{1}{\left(t' + \frac{G}{F}\right)\left(t' + \frac{\partial G/\partial r'}{\partial F/\partial r'}\right)}, \quad G = \sqrt{r'^3}, \quad F = -\frac{3}{2}\sqrt{2M}\sqrt{\frac{r'^3}{r_g'^3}} \ .
$$

Since

$$
\frac{G}{F} = \frac{\partial G/\partial r'}{\partial F/\partial r'}
$$

one obtains respecting (2.3)

$$
\kappa\mu_0 = \frac{4}{3}\frac{1}{\left(t' + \frac{G}{F}\right)^2}, \quad \frac{G}{F} = -\frac{2}{3}\sqrt{\frac{r_g'^3}{2M}} = -\frac{\rho_g'}{3} \ .
$$

With (2.4) one lastly gets

$$
\kappa\mu_0 = \frac{3}{\mathcal{R}_g^2} \ . \tag{3.11}
$$

The expression for the energy density is formally that of the interior Schwarzschild solution. At the beginning of the collapse ($t' = 0$) one has

$$
\Lambda(0) = 1, \quad r_g = r_g' \ .
$$

The surface of the stellar object is located at infinity ($r_g = \infty$) as we have pointed out in paper [1]. For (3.11) one can also write

$$\kappa\mu_0 = 3\frac{2M}{r_g^3} \ .$$

Thus, the infinitely large object at $t' = 0$ has vanishing mass density

$$\mu_0(0) = 0 \ .$$

As a consequence of

$$\kappa = \frac{8\pi k}{c^4}, \quad M = m\frac{k}{c^4}$$

the relation

$$m = \frac{4\pi}{3} r_g^3 \mu_0 \tag{3.12}$$

is valid, where m is the mass of the object which is enclosed by the sphere with the radius r_g. The conservation law leads to

$$T^{4'n'}{}_{||n'} = \mu_{0|4'} + A_{4'}\mu_0 = 0, \quad A_{4'} = U_{4'} + B_{4'} + C_{4'} = 3\frac{i}{\mathcal{R}_g}, \quad \partial_{4'} = \frac{\partial}{i\partial t'} \ . \tag{3.13}$$

Since

$$\mu_{0|m'} = \left\{0,0,0,-3i\frac{\mu_o}{\mathcal{R}_g}\right\}, \quad \mu_{0|m} = -3i\frac{\mu_o}{\mathcal{R}_g}{}'u_m \tag{3.14}$$

(3.13) is satisfied with (3.11).

4. FIELD EQUATIONS, NON-COMOVING SYSTEM

It is much more tedious to compute the field equation in the non-comoving system. One has to start with the static 4-bein and has to calculate the field strengths in a similar way as we have done for the comoving system. The 4-bein we have discussed in paper [1]. It is

$$\overset{1}{e}_1 = \alpha, \quad \overset{2}{e}_2 = r, \quad \overset{3}{e}_3 = r\sin\vartheta, \quad \overset{4}{e}_4 = \alpha\sqrt{\frac{r_g}{2M}\frac{y-1}{y^{3/2}}} \ , \tag{4.1}$$

$$y = \frac{1}{2}\left[\left(\frac{r'}{r'_g}\right)^2 - 1\right] + \frac{r'_g}{2M}\frac{r}{r'}, \quad dy = \frac{r'}{r'^2_g}dr' - \frac{1}{2M}\sqrt{\frac{2M}{r_g}}dt' \ . \tag{4.2}$$

Reading from the last expression the values

$$y_{|1'} = \underset{1'}{e}^{1'}\frac{\partial y}{\partial r'}, \quad y_{|4'} = \underset{4'}{e}^{4'}\frac{\partial y}{i\partial t'}$$

and subjecting them to the Lorentz transformation

$$L_{1'}^{1}=\alpha,\quad L_{4'}^{1}=-i\alpha v,\quad L_{1'}^{4}=i\alpha v,\quad L_{4'}^{4}=\alpha \tag{4.3}$$

with the parameters (2.6) one gets

$$y_{|1}=0\ ,$$

which facilitates the subsequent calculations considerably. It turns out that a negative quantity is included in the field strengths, calculated from (4.1), last expression. It is very similar to the Schwarzschild gravitational force. It is

$$E_1=\frac{1}{\sqrt{\frac{r_g}{2M}}}\left(\sqrt{\frac{r_g}{2M}}\right)_{|1}=\alpha v\frac{1}{\rho_g}\ . \tag{4.4}$$

If we define the quantity

$$H_m=\frac{1}{\alpha}\alpha_{|m}=\left\{E_1-3\alpha^3 v\frac{1}{\rho_g},0,0,-3i\alpha^3 v^2\frac{1}{\rho_g}\right\} \tag{4.5}$$

and if we fall back to (2.10) and (2.15) for its calculation, we finally obtain

$$A_{14}{}^{1}=H_4,\quad A_{41}{}^{4}=G_1=-E_1+H_1=-\alpha v\frac{1}{\rho_g}-3\alpha^3 v^3\frac{1}{\rho_g}\ . \tag{4.6}$$

The lateral field quantities

$$B_m=\left\{\frac{1}{\alpha r},0,0,0\right\},\quad C_m=\left\{\frac{1}{\alpha r},\frac{1}{r}\cot\vartheta,0,0\right\} \tag{4.7}$$

can easely be calculated with (4.1). Alternatively, we get the quantities from (3.2) with

$$B_m=L_m^{m'}B_{m'},\quad C_m=L_m^{m'}C_{m'}\ . \tag{4.8}$$

However, in general the field quantities transform inhomogeneously

$$A_{mn}{}^{s}=L_{m\ n\ s'}^{m'n's}{}'A_{m'n'}{}^{s'}+L_{s'}^{s}L_{n|m}^{s'}\ .$$

Putting

$$U_m=\{0,0,0,H_4\}\ , \tag{4.9}$$

the field equations in the static system take a form similar to (3.8), but are enriched by an expression containing the acceleration G

$$\begin{aligned}R_{mn}=&-h_{mn}\left[U^{s}{}_{\underset{1}{||s}}+U^{s}U_s\right]\\&-\left[B_{n\underset{2}{||}m}+B_nB_m\right]-b_nb_m\left[B^{s}{}_{\underset{2}{||s}}+B^sB_s\right]\\&-\left[C_{n\underset{3}{||}m}+C_nC_m\right]-c_nc_m\left[C^{s}{}_{\underset{3}{||s}}+C^sC_s\right]\\&-\left[G_{n\underset{4}{||}\underline{m}}+G_nG_m\right]-u_nu_m\left[G^{s}{}_{\underset{4}{||s}}+G^sG_s\right]\end{aligned} \tag{4.10}$$

wherein the graded derivatives are constructed similarly to (3.6)

$$U_{m||n \atop 1} = U_{m|n}, \quad B_{m||n \atop 2} = B_{m|n} - U_{nm}{}^{s}B_s, \quad C_{m||n \atop 3} = C_{m|n} - U_{nm}{}^{s}C_s - B_{nm}{}^{s}C_s$$
$$G_{m||\underline{n} \atop 4} = G_{m|\underline{n}} - B_{nm}{}^{s}G_s - C_{nm}{}^{s}G_s \quad . \tag{4.11}$$

The underbars denote the purely spatial derivatives.

With the Lorentz transformation

$$T_{mn} = L_{m\ n}^{m'n'} T_{m'n'} \tag{4.12}$$

we obtain the components of the stress-energy-momentum tensor for the static system

$$T_{11} = -\alpha^2 v^2 \mu_0, \quad T_{14} = -i\alpha^2 v \mu_0, \quad T_{44} = \alpha^2 \mu_0 \ . \tag{4.13}$$

With these expressions and with the useful auxiliary relations

$$B_{1|4} + B_1 U_4 = 0, \quad \left[G_{1|1} + G_1 G_1\right] + \left[U_{4|4} + U_4 U_4\right] = \frac{2}{\rho_g^2}$$

one can verify that the field equations are satisfied. The exterior solution, which matches the interior solution also in the non-comoving system at the surface of the stellar object need not be discussed. The problem has been treated in previous papers in detail, both the system in rest and the comoving system.

The contraction speed on the surface of the stellar object is

$$v_g = -\frac{r_g}{\mathcal{R}_g} = -\sqrt{\frac{2M}{r_g}}$$

in accordance with (2.6). This is the velocity of free fall in the Schwarzschild field for an observer coming from the infinite. v_g reaches the speed of light at $r_g = 2M$. The corresponding Lorentz factor α_g is infinite at this location. Both quantities v_g and α_g enter into the field strengths (4.4) and (4.5). Thus, these forces are infinite at the event horizon. Consequently, the OS-model has an event horizon at $r_g = 2M$. OS have given the non-comoving co-ordinate time of the exterior as

$$t = \frac{3}{2}\frac{1}{\sqrt{2M}}\left(r'^{3/2} - r^{3/2}\right) - 2\sqrt{2Mr} - 2M \ln\frac{\sqrt{r} - \sqrt{2M}}{\sqrt{r} + \sqrt{2M}} \ .$$

It is the time the surface of the star, coming from infinity, needs to reach the position r. To reach the event horizon at $r = 2M$ the time t will be infinite. The same applies to the proper time of falling objects, as we have shown earlier [3-7] for arbitrary objects. Thus, the OS-star can never shrink to the event horizon. The formation of a black hole is not possible.

5. SUMMARY

We have shown in two papers that the surface of the collapsing OS star has the same fate as any object being in free fall in the Schwarzschild field. The star's surface reaches asymptotically the event horizon with the velocity of light in infinite co-ordinate time and infinite proper time as well. At this location the gravitational field strengths blow up. Moreover, the stellar object is infinitely large and its energy density zero at the beginning of the collapse. Although the OS model is mathematically correct it cannot represent a physical object and cannot serve for a black hole.

6. REFERENCES

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