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FIELD EQUATIONS FOR EINSTEIN'S ELEVATOR

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Contents

1. Introduction 2
2. Preliminaries 2
3. Field strengths and field equations 4
4. Summary.. 6
5. References... 6

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Abstract: We are studying the field equations for observers freely falling from an arbitrary position in the Schwarzschild field. The field strengths referring to the accelerated system are connected to those of the system in rest by a Lorentz transformation.

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1. INTRODUCTION

In former papers [1-6] we have extensively studied the free fall in the Schwarzschild field with particular attention to the case of an arbitrary distant position with respect to the central mass. Since we have significantly deviated from the standard treatment in the literature, we want to ensure our results by constructing the field equations for the free-falling system.

2. PRELIMINARIES

In [3] we have determined the 4-bein system for an observer B' who falls away from the position r_0

$$\overset{1'}{e}_{1'} = -\frac{\alpha' v'}{\alpha}, \quad \overset{2'}{e}_{2'} = r, \quad \overset{3'}{e}_{3'} = r\sin\vartheta, \quad \overset{4'}{e}_{4'} = \frac{\alpha'}{\alpha}. \tag{2.1}$$

The co-ordinates $\{r', t'\}$ are comoving co-ordinates. In the non-moving system B with $\{r, t\}$ the usual Schwarzschild values

$$\overset{1}{e}_{1} = \alpha, \quad \overset{2}{e}_{2} = r, \quad \overset{3}{e}_{3} = r\sin\vartheta, \quad \overset{4}{e}_{4} = a, \quad a = 1/\alpha = \sqrt{1-2M/r} \tag{2.2}$$

are valid. The Lorentz transformation mediates between the two systems

$$L^{1}_{1'} = \alpha', \quad L^{1}_{4'} = -i\alpha' v', \quad L^{4}_{1'} = i\alpha' v', \quad L^{4}_{4'} = \alpha' \;, \tag{2.3}$$

whereby the fall velocity v' of B' is according to

$$v' = v(r, r_0) = \frac{-\sqrt{\frac{2M}{r}} - \left(-\sqrt{\frac{2M}{r_0}}\right)}{1 - \sqrt{\frac{2M}{r}}\sqrt{\frac{2M}{r_0}}} \tag{2.4}$$

composed of the two velocities $v = -\sqrt{2M/r}$ and $v_0 = -\sqrt{2M/r_0}$, taking into account Einstein's addition law of velocities and α' is the associated Lorentz factor. Between the co-ordinates the Lemaître transformation specified by

$$\begin{aligned} &\Lambda^{1}_{1'} = -\frac{\alpha'^2}{\alpha^2} v', \quad \Lambda^{1}_{4'} = -i\frac{\alpha'^2}{\alpha^2} v', \quad \Lambda^{4}_{1'} = -i\alpha'^2 v'^2, \quad \Lambda^{4}_{4'} = \alpha'^2 \\ &\Lambda^{1'}_{1} = -\frac{\alpha^2}{v'}, \qquad \Lambda^{1'}_{4} = -i, \qquad \Lambda^{4'}_{1} = -i\alpha^2 v', \qquad \Lambda^{4'}_{4} = 1 \end{aligned} \tag{2.5}$$

mediates.

For further calculations we need the derivatives of r in the tetrad representation

$$r_{|m} = \{1,0,0,0\}a, \quad r_{|m'} = \{\alpha',0,0,-i\alpha'v'\}a\,, \tag{2.6}$$

wherein the first relation is calculated with (2.2) and the second is obtained with (2.3) from the first. We note that $r_{|4} = 0$. v is the velocity of a freely falling observer B'' who comes from infinity. We have

$$v_{|m} = \{1,0,0,0\}\frac{a}{\rho}, \quad v_{|m'} = \{\alpha',0,0,-i\alpha'v'\}\frac{a}{\rho}\,. \tag{2.7}$$

We have derived the second term with the Lorentz transformation (2.3). The quantity

$$\rho = \sqrt{\frac{2r^3}{M}} \tag{2.8}$$

is the radius of curvature of the Schwarzschild parabola. v_0 is the velocity of a freely falling observer who comes from infinity, measured at the position r_0 ie at the position the observer whose laws we want to investigate, falls off. v_0 is used as a reference velocity and is a constant.

From the addition law of velocities

$$v' = \frac{v - v_0}{1 - vv_o} \tag{2.9}$$

we obtain with (2.7)

$$v'_{|m} = \{1,0,0,0\}\frac{\alpha}{\alpha'^2}\frac{1}{\rho}, \quad v'_{|m'} = \{\alpha',0,0,-i\alpha'v'\}\frac{\alpha}{\alpha'^2}\frac{1}{\rho}\,. \tag{2.10}$$

The $\alpha, \alpha_0, \alpha'$ are the Lorentz factors associated with the velocities v, v_0, v'. With

$$\frac{1}{\alpha}d\alpha = \alpha^2 v dv$$

we obtain

$$\begin{aligned} \frac{1}{\alpha}\alpha_{|m} &= \{1,0,0,0\}\alpha v\frac{1}{\rho}, \quad \frac{1}{\alpha}\alpha_{|m'} = \{\alpha',0,0,-i\alpha'v'\}\alpha v\frac{1}{\rho} \\ \frac{1}{\alpha'}\alpha'_{|m} &= \{1,0,0,0\}\alpha v'\frac{1}{\rho}, \quad \frac{1}{\alpha'}\alpha'_{|m'} = \{\alpha',0,0,-i\alpha'v'\}\alpha v'\frac{1}{\rho} \end{aligned}\,. \tag{2.11}$$

With this set of equations we are able to calculate the field quantities for the free-falling observer B' and the observer at rest B. To configure compact field equations for these observers, we apply here in deviation from previous papers the geometrical quantities U which have the opposite sign in contrast to the physical quantities E: $U = -E$.

3. FIELD STRENGTHS AND FIELD EQUATIONS

Having derived the basic relations, the transformations of co-ordinate and reference systems, the velocities, and their change, we are armed to calculate the field strengths and the field equations of the freely falling system.

From

$$
\begin{aligned}
U_1 = U_{41}{}^4 = -\overset{4}{e}_4 \underset{4|1}{e}{}^4, \quad U_4 = U_{14}{}^1 = -\overset{1}{e}_1 \underset{1|4}{e}{}^1 \\
{}'U_{1'} = {}'U_{4'1'}{}^{4'} = -\overset{4'}{e}_{4'} \underset{4'|1}{e}{}^{4'}, \quad {}'U_{4'} = {}'U_{1'4'}{}^{1'} = -\overset{1'}{e}_{1'} \underset{1'|4}{e}{}^{1'}
\end{aligned}
\tag{3.1}
$$

we calculate the field quantities

$$
U_m = \{1,0,0,0\}\left(-\alpha v \frac{1}{\rho}\right), \quad {}'U_{m'} = \{-i\alpha_0 v_0, 0, 0, \alpha_0\}\left(-\frac{i}{\rho}\right) . \tag{3.2}
$$

The relationship of these quantities is obtained with the inhomogeneous transformation law of the Ricci-rotation coefficients

$$
{}'A_{m'n'}{}^{s'} = L^{m\ n\ s'}_{m'n's} A_{mn}{}^s + L^{s'}_s L^s_{n'|m'}, \quad {}'L_{n'} = {}'L_{s'n'}{}^{s'} . \tag{3.3}
$$

The last term we call Lorentz term[1] and we write it as

$$
{}'L_{m'n'}{}^{s'} = L^{s'}_s L^s_{n'|m'} = h_{m'}{}^{s'}\,{}'L_{n'} - h_{m'n'}\,{}'L^{s'} . \tag{3.4}
$$

Therein

$$
h_{m'n'} = \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 1 \end{pmatrix} \tag{3.5}
$$

is a submatrix of the metric in tetrad representation. For the inverse transformation we form analogous to (3.4)

$$
L_{mn}{}^s = L^s_{s'} L^{s'}_{n|m} = h_m{}^s L_n - h_{mn} L^s, \quad L_n = L_{sn}{}^s . \tag{3.6}
$$

For the Lorentz terms one obtains

$$
{}'L_{1'} = i\alpha'^2 v'_{|4'}, \quad {}'L_{4'} = -i\alpha'^2 v'_{|1'}, \quad L_1 = -i\alpha'^2 v'_{|4}, \quad L_4 = i\alpha'^2 v'_{|1} . \tag{3.7}
$$

With (2.10) this gives

$$
{}'L_{1'} = \alpha' v' \alpha \frac{1}{\rho}, \quad {}'L_{4'} = -i\alpha' \alpha \frac{1}{\rho}, \quad L_1 = 0, \quad L_4 = i\alpha \frac{1}{\rho} . \tag{3.8}
$$

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[1] In our previous papers on the equations of motion the dynamic quantity L has been called counterforce G.

We also recognize that

$$'L_{m'} = -L^{m}_{m'} L_m \tag{3.9}$$

and we find with (3.2)

$$\begin{gathered} 'U_{m'} = U_{m'} + 'L_{m'} \quad U_m = 'U_m + L_m, \quad U_{m'} = L^{m}_{m'} U_m \quad 'U_m = L^{m'}_{m} 'U_{m'} \\ U_m = \{1,0,0,0\}\left(-\alpha v \frac{1}{\rho}\right) \end{gathered} \quad . \tag{3.10}$$

$E_m = -U_m$ is the force of gravity in the static Schwarzschild field.

We calculate the lateral field quantities B and C for the free-falling system with

$$\begin{aligned} B_{m'} &= 'A_{2'm'}{}^{2'} = -\overset{2'}{e}_{2'} \underset{2'|m'}{e^{2'}} = \{\alpha',0,0,-i\alpha' v'\}\frac{a}{r} \\ C_{m'} &= 'A_{3'm'}{}^{3'} = -\overset{3'}{e}_{3'} \underset{3'|m'}{e^{3'}} = \left\{\alpha'\frac{a}{r}, \frac{1}{r}\cot\vartheta, 0, -i\alpha' v'\frac{a}{r}\right\} \end{aligned} \tag{3.11}$$

and we also recognize that the lateral field quantities transform as vectors

$$B_{m'} = L^{m}_{m'} B_m, \quad C_{m'} = L^{m}_{m'} C_m \ , \tag{3.12}$$

whereby are the

$$B_m = \left\{\frac{a}{r},0,0,0\right\}, \quad C_m = \left\{\frac{a}{r},\frac{1}{r}\cot\vartheta,0,0\right\} \tag{3.13}$$

the well-known Schwarzschild values. With the unit vectors in the 2- and 3-direction

$$b_{m'} = \{0,1,0,0\}, \quad c_{m'} = \{0,0,1,0\} \tag{3.14}$$

we finally have

$$B_{m'n'}{}^{s'} = b_{m'} B_{n'} b^{n'} - b_{m'} b_{n'} B^{n'}, \quad C_{m'n'}{}^{s'} = c_{m'} C_{n'} c^{n'} - c_{m'} c_{n'} C^{n'} \tag{3.15}$$

and we can write down the complete Ricci-rotation coefficients for the freely falling system as

$$'A_{m'n'}{}^{s'} = 'U_{m'n'}{}^{s'} + B_{m'n'}{}^{s'} + C_{m'n'}{}^{s'} \ . \tag{3.16}$$

If we further define the graded derivatives as

$$'U_{\underset{1}{n\|m}} = 'U_{n|m}, \quad B_{\underset{2}{n'\|m'}} = B_{n'|m'} - 'U_{m'n'}{}^{s'} B_{s'}, \quad C_{\underset{3}{n'\|m'}} = C_{n'|m'} - B_{m'n'}{}^{s'} C_{s'} - 'U_{m'n'}{}^{s'} C_{s'} \ , \tag{3.17}$$

we obtain the Ricci in a compact form

$$R_{m'n'} = -\left[{'U^{s'}}_{\underset{1}{\|}s'} + 'U^{s'}\,'U_{s'} \right] h_{m'n'}$$
$$- \left[B_{n'\underset{2}{\|}m'} + B_{n'}B_{m'} \right] - b_{n'}b_{m'} \left[{B^{s'}}_{\underset{2}{\|}s'} + B^{s'}B_{s'} \right] . \qquad (3.18)$$
$$- \left[C_{n'\underset{3}{\|}m'} + C_{n'}C_{m'} \right] - c_{n'}c_{m'} \left[{C^{s'}}_{\underset{3}{\|}s'} + C^{s'}C_{s'} \right]$$

If we apply this to the particular subequations one has

$$'{U^{s'}}_{\underset{1}{\|}s'} + 'U^{s'}\,'U_{s'} = -\frac{4}{\rho^2}$$
$$B_{n'\underset{2}{\|}m'} + B_{n'}B_{m'} = h_{m'n'}\frac{2}{\rho^2}, \quad {B^{s'}}_{\underset{2}{\|}s'} + B^{s'}B_{s'} = \frac{4}{\rho^2} \qquad . \qquad (3.19)$$
$$C_{n'\underset{3}{\|}m'} + C_{n'}C_{m'} = h_{m'n'}\frac{2}{\rho^2} - b_{n'}b_{m'}\frac{4}{\rho^2}, \quad {C^{s'}}_{\underset{3}{\|}s'} + C^{s'}C_{s'} = 0$$

Thus one finds $R_{m'n'} = 0$ the vacuum field equations of the Schwarzschild theory in a reference system connected with an observer freely falling from an arbitrary position.

4. SUMMARY

Einstein's field equations are satisfied with the field quantities derived by us from the tetrad system (2.1). The Ricci is Lorentz invariant as expected. The field equations of the freely falling system can be derived by a Lorentz transformation from those of the static system.

5. REFERENCES

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