ARG Austrian Reports on Gravitation

ARG-2011-01

FREE FALL IN THE SCHWARZSCHILD FIELD

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Keywords: free fall from any position, velocity, time function

Contents

The fall time of an object being released from infinity and an arbitrary position, respectively, in the Schwarzschild field are considered. The calculations are performed in Schwarzschild-standard co-ordinates and Einstein-Rosen co-ordinates.

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1. INTRODUCTION

In the last decades the problem of free fall has repeatedly been treated in the literature. Although there is general agreement that an observer who comes from infinity, reaches the event horizon with the velocity of light, it came to controversial views regarding the speed of an infalling observer who falls from a finite position to the center of gravity. We will again tackle the problem and we will show that an observer who comes from infinity or from any other position would reach the event horizon with the speed of light and therefore would need an infinite proper time. To overcome the problem, the knowledge of 1915 is sufficient. We only need the Einstein addition theorem of velocities and the velocity formula for the free fall in the Schwarzschild field.

2. VELOCITIES IN THE SCHWARZSCHILD FIELD

The actual problem consists in the fact that initially only the speed of an observer who comes from the infinite is known. It is determined by the Schwarzschild geometry as

$$v = v(r) = -\sqrt{\frac{2M}{r}} \quad . \tag{2.1}$$

The velocity of an observer who falls from an arbitrary point can only be determined by a more complicated method. For this purpose we consider the following:

An object coming from infinity has at an arbitrary point r_0 the velocity $v_0 = -\sqrt{2M/r_0}$. Another object is released from this point at the moment when the first object is passing this position. The difference in their fall velocity is v_0 at this very moment. The difference decreases during the fall according to Einstein's composition law of velocities. The speed of the second object with regard to the static Schwarzschild system is calculated according to the relative velocity of the first object

$$v' = v(r, r_0) = \frac{-\sqrt{\frac{2M}{r}} - \left(-\sqrt{\frac{2M}{r_0}}\right)}{1 - \sqrt{\frac{2M}{r}}\sqrt{\frac{2M}{r_0}}} \quad . \tag{2.2}$$

At the starting position one has $v(r_0, r_0) = 0$, at the event horizon $v(2M, r_0) = -1$. Fig. 1 shows some examples.

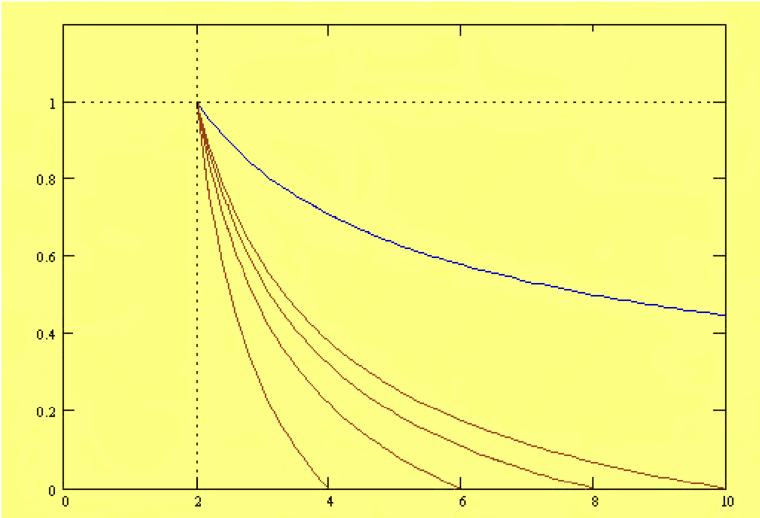

Fig.1

The top curve corresponds to the observer who comes from the infinite. In order to handle the velocity relations we introduce three reference systems: B being at rest in the Schwarzschild field, B' accompanying an observer with the speed v', and B" freely falling from infinity with the speed v. The systems are connected by the Lorentz relations

$$v' = \frac{v - v_0}{1 - v v_0}, \qquad v = \frac{v' + v_0}{1 + v' v_0}, \qquad v_0 = \frac{v - v'}{1 - v v'} \ , \tag{2.3}$$

$$\alpha' = \alpha\alpha_0 (1 - v v_0), \quad \alpha = \alpha'\alpha_0 (1 + v' v_0), \quad \alpha_0 = \alpha'\alpha (1 - v' v) \ , \tag{2.4}$$

$$\alpha' v' = \alpha\alpha_0 (v - v_0), \quad \alpha v = \alpha'\alpha_0 (v' + v_0), \quad \alpha_0 v_0 = \alpha\alpha' (v - v') \ . \tag{2.5}$$

The observer B" does not change his position in the comoving system. Therefore, one has

$$\frac{dx"}{dT"} = 0, \qquad x" = \text{const.} \ , \tag{2.6}$$

where T" is the proper time of B". In view of the system B the velocity of B" is

$$\frac{dx}{dT} = v \ , \tag{2.7}$$

if we use the proper time T of the static system. Applying the well-known relation

$$\frac{dT}{dT"} = \alpha \tag{2.8}$$

with α as the Lorentz factor of the transformation $x \leftrightarrow x"$ and by taking into consideration the relation $dx = \alpha dr$, one has

$$v = \frac{dr}{dT"} \ . \tag{2.9}$$

The integral

$$T''(r) = -\int \sqrt{\frac{r}{2M}} dr = -\frac{1}{3}\sqrt{\frac{2r^3}{M}} \tag{2.10}$$

is an expression well known in the literature. It shows graphically a curve that runs from $r = 0$ to infinity. The function provides for any point the time which the observer needs starting from $r = 0$ to reach that point. Since there is invariance under time reversal, one obtains for the fall time a function which is infinit at $r = 0$.

It should be noted that the curve of $T''(r)$ crosses the event horizon, although an infalling body would always reach the speed of light at the event horizon. The fall velocity

$$v(r) = -\sqrt{\frac{2M}{r}} \tag{2.11}$$

can, mathematically, be extended into the inner region $0 < r < 2M$ of the Schwarzschild solution. Thus, it is mathematically quite correct that the integral (2.10) also covers the inner region.

Since physics forces us to exclude the inner region from the integration process, we use instead the standard Schwarzschild co-ordinates, the Einstein-Rosen co-ordinates[1] [1]. From the equation of the Schwarzschild parabola

$$R^2 = 8M(r - 2M)$$

results

$$r = \frac{R^2 + 16M^2}{8M}, \tag{2.12}$$

wherewith we obtain the singularity-free line element

$$ds^2 = \frac{R^2 + 16M^2}{16M^2} dR^2 + \left(\frac{R^2 + 16M^2}{8M}\right)^2 \left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right) - \frac{R^2}{R^2 + 16M^2} dt^2 \ . \tag{2.13}$$

$R = 0$ is the vertex of the Schwarzschild parabola and corresponds to $r = 2M$. For $R = 0$ one obtains $dx = dR$, the tangent of the Schwarzschild parabola at the vertex. On principle R cannot take values in the inner region of the Schwarzschild metric.

It should be noted that neither r nor R corresponds to a physical distance. The physical distance from the horizon we get from

$$r^* = \int_{2M}^{r} dx = \int_{2M}^{r} \alpha dr = r\sqrt{1 - \frac{2M}{r}} + M \ln\left(\frac{1 + \sqrt{1 - \frac{2M}{r}}}{1 - \sqrt{1 - \frac{2M}{r}}}\right) . \tag{2.14}$$

This is the rectification formula for the Schwarzschild parabola. r and R are both auxiliary variables which define Cartesian co-ordinates in the higher dimensional embedding space. Both can be used to formulate the theory of Schwarzschild. For the velocity of free fall results

[1] The co-ordinates we use differ from the original Einstein-Rosen co-ordinates by the factor $\sqrt{8M}$.

$$v(R) = -\frac{4M}{\sqrt{R^2 + 16M^2}} \tag{2.15}$$

and has the value -1 for $R = 0$. With

$$dr = \frac{R}{4M} dR$$

one first obtains by integration of $dT'' = \frac{1}{v} dr$

$$f(R) = \int \frac{R}{16M^2} \sqrt{R^2 + 16M^2} dR = \frac{1}{48M^2} \sqrt{\left(R^2 + 16M^2\right)^3} + C \ . \tag{2.16}$$

If we choose the integration constant in such a way that we obtain for $R = 0$ also $T'' = 0$ we finally have

$$T''(R) = f(R) - f(0) \ , \tag{2.17}$$

a curve, starting at $R = 0$, $(r = 2M)$ and growing to the infinite. Time reversal results in the image that an observer incoming with velocity v(R) from infinity, takes an infinit time to reach the event horizon which he can never cross.

It is more difficult to compute the time function for observers that are incoming from an arbitrary position r_0. We have to use the relations

$$\frac{dx}{dT} = v', \quad \frac{dT}{dT'} = \alpha', \quad x' = \text{const.} \ . \tag{2.18}$$

With

$$\frac{\alpha dr}{dT'} = \alpha' v'$$

one has

$$dT' = \frac{\alpha}{\alpha' v'} dr = \frac{1}{\alpha_0 \left(v - v_0\right)} dr \ . \tag{2.19}$$

Integrating this expression one cannot prevent that r runs beneath the event horizon. Therefore, we recall the Einstein-Rosen co-ordinates. One has

$$dT' = \frac{R_0}{\sqrt{R_0^2 + 16M^2}} \frac{1}{\frac{4M}{\sqrt{R_0^2 + 16M^2}} - \frac{4M}{\sqrt{R^2 + 16M^2}}} \frac{R}{4M} dR \ .$$

After some rearrangement one obtains an integral of the type

$$\int \frac{\sqrt{x}}{\sqrt{x} - 1} dx = x + 2\sqrt{x} + 2\ln(1 - \sqrt{x}), \quad x < 1, \quad \lim_{x \to 1} \int \frac{\sqrt{x}}{\sqrt{x} - 1} dx = \infty \ .$$

For $x = \frac{R^2 + 16M^2}{R_0^2 + 16M^2}$ one gets the rise time $f(R)$. Starting up from $R = 0$ and ending at R_0 it increases to infinity. In this case it is simple to mirror the function in such a way that an observer starting from R_0 reaches the event horizon at $R=0$ after an infinitely long time. Replacing in the time function the variable R by R_0-R the observer starts at $R=R_0$ and passes through the fall distance at $R=0$. One has

$$f(R,R_0)=\frac{R_0}{32M^2}\left(R_0^2+16M^2\right)\times \times\left[\frac{(R_0-R)^2+16M^2}{R_0^2+16M^2}+2\sqrt{\frac{(R_0-R)^2+16M^2}{R_0^2+16M^2}}+2\ln\left(1-\sqrt{\frac{(R_0-R)^2+16M^2}{R_0^2+16M^2}}\right)\right]+C \quad . \tag{2.20}$$

After a suitable choice of the integration constant

$$T'(R,R_0)=f(R,R_0)-f(R_0,R_0) \tag{2.21}$$

one gains the following plot

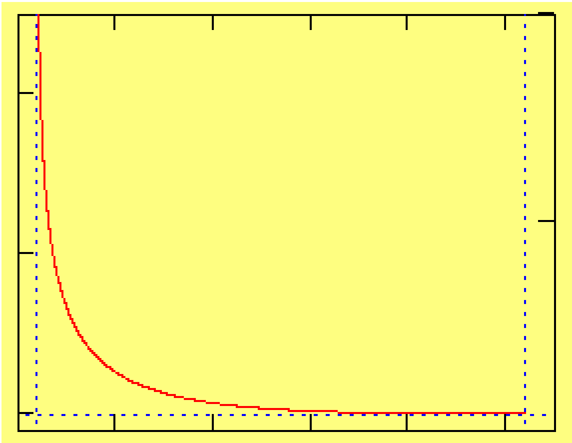

Fig.1.

In Fig. 1 marks the left vertical line the event horizon. It turns out that no object infalling from a finite or infinite position can reach the event horizon in finite proper time.

From the previous considerations conclusions for a collapse of star can be drawn. The surface of a stellar object is located at the position r_0, thus, even if one assumes the maximum speed of contraction, namely the free-fall, the object can never contract to the event horizon, or even fall below it. This will have far-reaching consequences for the theory of stellar collapse. Models that satisfy this condition have been proposed by Mitra [2-5].

3. OUTLOOK

Having calculated the velocities of an observer freely falling from an arbitrary position to a stellar object, we are able to investigate the accelerations and gravitational field strength acting on this observer. We will publish this elsewhere.

4. REFERENCES

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