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TIDAL FORCES AND SCHWARZSCHILD FIELD

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Contents

Tidal forces act on an extended body in the static Schwarzschild field and on a freely falling observer as well. We will geometrically explain these forces as second fundamental forms of surfaces and we will work out the relation between the two kinds of tidal forces.

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1. INTRODUCTION

Tidal forces are acting on any body being exposed to the static Schwarzschild field. They cause a radial stretching and a lateral compression of the body. Misner, Thorn, and Wheeler (p. 861) [1] have calculated these effects for an extended body. Sharon [2] has drawn the field lines for the tidal forces. The static Einstein field equations do not exhibit tidal forces. Thus, they are derived from the Riemann in the literature. However, we will show that one can obtain the tidal forces from the subequations of the Einstein field equations. In former papers [3,4] we have calculated the tidal forces also for a freely falling system, and we have set up the field equations for these forces. In the present paper we will show how the two kinds of tidal forces are related. We will explain the tidal forces of the static system as the second fundamental forms of a surface embedded in a higher dimensional flat space and the tidal forces felt by freely falling observers as the second fundamental forms of a shrinking space.

2. TIDAL FORCES OF THE STATIC SYSTEM

In the literature, tidal forces are calculated from the Riemann tensor. A lot of work has been done by Mashhoon [5], and Theiss [6], Chicone and Mashhoon [7-9], and Costa and Herdeiro [10,11]. Synge [12] has shown that the components of the Riemann

$$R_{2112} = R_{3113} = R_{4224} = R_{4334} = -\frac{M}{r^3}, \quad R_{4114} = R_{3223} = \frac{2M}{r^3} \tag{2.1}$$

can be attributed to the deviation of geodesics by

$$\eta^s{}_{||mn} u^m u^n = -R_{rmn}{}^s u^m u^n \eta^r, \quad \frac{D^2\eta^s}{D\tau^2} = -R_{rmn}{}^s u^m u^n \eta^r, \tag{2.2}$$

wherein η is the deviation vector of two adjacent geodesics, dependent on the distance of these geodesics. (2.2) describes the relative acceleration of two points on the geodesics.

Unfortunately, the representation (2.1) hides the geometrical background. From the Riemann in tetrad representation

$$R_{smn}{}^r = A_{mn}{}^r{}_{|s} - A_{sn}{}^r{}_{|m} + A_{mn}{}^h A_{sh}{}^r - A_{sn}{}^h A_{mh}{}^r + A_{ms}{}^h A_{hn}{}^r - A_{sm}{}^h A_{hn}{}^r \tag{2.3}$$

follows

$$R_{2112} = R_{3113} = R_{4224} = R_{4334} = \frac{1}{\rho}\frac{1}{r}\sin\varepsilon, \quad R_{4114} = \frac{1}{\rho^2} - E_1\frac{1}{\rho}\rho_{|1}, \quad R_{3223} = \frac{1}{r^2}\sin^2\varepsilon\ . \tag{2.4}$$

Therein is

$$\sin\varepsilon = v = -\sqrt{\frac{2M}{r}}, \quad \cos\varepsilon = a = \sqrt{1-\frac{2M}{r}}, \quad \rho = \sqrt{\frac{2r^3}{M}}, \quad E_1 = \frac{1}{\rho}\frac{v}{a}, \quad E_1\frac{1}{\rho}\rho_{|1} = -\frac{3}{\rho^2} \tag{2.5}$$

with the angle of ascent ε and the curvature radius ρ of the Schwarzschild parabola. Apart from the last term in (2.5), which we have still to explain, the components of the Riemann are products of the curvatures of the normal and inclined slices of a surface embedded in a 5-dimensional flat space. That an embedding into 5-dimensions is possible without violating the theorems of Kasner and Eisenhart we have shown in previous papers [13,14]. The parabolic Schwarzschild geometry is derived by projection techniques from a 4-dimensional pseudo-hyper sphere embedded in a 5-dimensional flat space and by the use of the theory of double surfaces developed by us. If the X are the connexion coefficients of the flat space, parameterized in pseudo-polar co-ordinates, A the connexion coefficients of the 5-dimensional parabolic geometry, and $\mathfrak{p}$ the projectors one has

$$\mathfrak{p}_{ab}^{gh} R_{ghc}{}^{d}(X) = {}^{5}R_{adc}{}^{d}(A) + 2A_{hc}{}^{d}\left(\mathfrak{p}^{-1}\right)_{f}^{h} \mathfrak{p}_{[a||b]}^{f} = 0 \ , \tag{2.6}$$

whereby the last term provides the above-mentioned expression $E_1 \frac{1}{\rho} \rho_{|1}$.

The 5-dimensional connexion coefficients contain the quantities

$$A_{11} = \frac{1}{\rho}, \quad A_{22} = \frac{v}{r}, \quad A_{33} = \frac{v}{r}, \quad A_{44} = \frac{1}{\rho} \tag{2.7}$$

which are the generalized second fundamental forms of the surface theory and are related to the tidal forces of the Schwarzschild geometry. Defining the unit matrix in the 2-dimensional [1,4]-subspace

$$h_{mn} = \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 1 \end{pmatrix}, \tag{2.8}$$

one gets for the Riemann and Ricci

$$R_{rmn}{}^{s} = 2A_{m[n}A_{r]}{}^{s} + 2h_{m[n}h_{r]}{}^{s}\frac{3}{\rho^2}, \quad R_{mn} = 2A_{m[n}A_{s]}{}^{s} + 2h_{m[n}h_{s]}{}^{s}\frac{3}{\rho^2} \ . \tag{2.9}$$

Apart from the second terms on the right sides these are the Gauss and the contracted Gauss equations. With (2.7) one recognizes that the Ricci vanishes. That means that we deal with vacuum field equations. From this follows that the tidal forces emerging from the Ricci of the static system are hidden by the very expression we have derived from the embedding. The Einstein tensor has the form

$$G_{mn} = \left[2A_{m[n}A_{s]}{}^{s} - g_{mn}A^{r}{}_{[r}A_{s]}{}^{s}\right] + \left[2h_{m[n}h_{s]}{}^{s} - g_{mn}h^{r}{}_{[r}h_{s]}{}^{s}\right]\frac{3}{\rho^2} \ , \tag{2.10}$$

where the second brackets can be simplified to $h_{mn} - g_{mn}$. If we calculate the divergence of the Einstein tensor

$$G_{m}{}^{n}{}_{||n} = \left[2A_{m}{}^{[n}A_{s}{}^{s]} - \delta_{m}^{n}A^{r}{}_{[r}A_{s]}{}^{s}\right]_{||n} + \left[\left(h_{m}{}^{n}{}_{||n} - \delta_{m||n}^{n}\right)\frac{3}{\rho^2} + \left(h_{m}{}^{n} - \delta_{m}^{n}\right)\left(\frac{3}{\rho^2}\right)_{||n}\right] , \tag{2.11}$$

then there remains from the second brackets only

$$\frac{3}{\rho^2}h_{m}{}^{n}{}_{||n} = \frac{4}{\rho^3}\rho_{|m}, \quad \rho_{|1} = -3\frac{a}{v} \ , \tag{2.12}$$

while the first brackets can be written as

$$A_{<n}{}^{n}A_{m}{}^{s}{}_{||s>}\ .$$

The contracted generalized Codazzi equations are contained in (2.11). Because of

$$2A_{[n}{}^{s}{}_{||m]} = \frac{2}{\rho^2}\rho_{||[n}u_{m]}u^s, \quad 2A_{[m}{}^{n}{}_{||n]} = \frac{1}{\rho^2}\rho_{|m} \tag{2.13}$$

they do not vanish. The reason is that the Schwarzschild geometry is not a V_4 which could be embedded into a flat 5-dimensional space. Inserting (2.12) and (2.13) into (2.11) one obtains the trivial result that the divergence of the Einstein tensor vanishes.

The tidal forces emerge more noticeably if one calculates the subequations of the Einstein field equations. In previous papers [13,14] we have decomposed the Ricci into its subequations which are the curvature equations of the geometry

$$\begin{aligned} R_{mn} = &-\left[B_{m\underset{2}{||}n} + B_mB_n\right] - b_mb_n\left[B^s{}_{\underset{2}{||}s} + B^sB_s\right] \\ &-\left[C_{m\underset{3}{||}n} + C_mC_n\right] - c_mc_n\left[C^s{}_{\underset{3}{||}s} + C^sC_s\right]\ . \\ &+\left[E_{m\underset{4}{||}n} - E_mE_n\right] + u_mu_n\left[E^s{}_{\underset{4}{||}s} - E^sE_s\right] \end{aligned} \tag{2.14}$$

The unit vectors of the 2,3,4-directions are b, c, and u. The curvature quantities are B, C, and E, where E is the force of gravity. The graded derivatives [13,14] are applied. For the individual brackets one obtains

$$\begin{aligned} &B_{m\underset{2}{||}n} + B_mB_n = A_{mn}A_{22}, && B^s{}_{\underset{2}{||}s} + B^rB_r = A_s{}^sA_{22}, && m,n,s = 1 \\ &C_{m\underset{3}{||}n} + C_mC_n = A_{mn}A_{33}, && C^s{}_{\underset{3}{||}s} + C^sC_s = A_s{}^sA_{33}, && m,n,s = 1,2 \quad . \\ &E_{m\underset{4}{||}n} - E_mE_n = A_{mn}A_{44} + \frac{3}{\rho^2}h_{mn}, && E^s{}_{\underset{4}{||}s} - E^sE_s = A_s{}^sA_{44} + \frac{3}{\rho^2}, && m,n,s = 1,2,3 \end{aligned} \tag{2.15}$$

Summing up these equations one gets again (2.9). The next step will be to set the tidal forces of the static system into relation to the ones of the freely falling system.

3. THE FORCES OF THE FREELY FALLING SYSTEM

We use the Lorentz transformation to the freely falling system

$$L^1_{1'} = \alpha, \quad L^1_{4'} = -i\alpha v, \quad L^4_{1'} = i\alpha v, \quad L^4_{4'} = \alpha, \quad \alpha = 1\big/\sqrt{1-v^2} \tag{3.1}$$

and the Lorentz term

$$L_{m'n'}{}^{s'} = L^{s'}_{s}L^{s}_{n'|m'} = h_{m'}{}^{s'}L_{n'} - h_{m'n'}L^{s'}, \quad L_{n'} = L_{s'n'}{}^{s'} = \left\{\alpha^2 v\frac{1}{\rho}, 0, 0, -i\alpha^2\frac{1}{\rho}\right\} \tag{3.2}$$

which emerges from the transformation of the covariant derivative of a vector

$$\Phi_{m'||n'} = L^{m\ n}_{m'n'}\Phi_{m||n} = \Phi_{m'|n'} - \left[A_{n'm'}{}^{s'} + L^{s'}_{s}L^{s}_{m'|n'}\right]\Phi_{s'}, \quad A_{n'm'}{}^{s'} = L^{n\ m\ s'}_{n'm's}A_{nm}{}^{s} \ . \tag{3.3}$$

The curvature quantities

$$B_n = \left\{\frac{a}{r},0,0,0\right\}, \quad C_n = \left\{\frac{a}{r},\frac{1}{r}\cot\vartheta,0,0\right\} \ , \tag{3.4}$$

transform like vectors

$$B_{n'} = \left\{\frac{1}{r},0,0,-iv\frac{1}{r}\right\}, \quad C_{n'} = \left\{\frac{1}{r},\frac{1}{r}\cot\vartheta,0,-iv\frac{1}{r}\right\} \ , \tag{3.5}$$

but the time-like part of the connexion coefficients

$$E_{mn}{}^{s} = -\left[h_m{}^{s}E_n - h_{mn}E^{s}\right], \quad E_n = \left\{\alpha v\frac{1}{\rho},0,0,0\right\} \tag{3.6}$$

transforms inhomogeneously

$$'E_{n'm'}{}^{s'} = E_{n'm'}{}^{s'} + L_{n'm'}{}^{s'}, \quad E_{n'm'}{}^{s'} = L^{n\ m\ s'}_{n'm's}E_{nm}{}^{s} \ . \tag{3.7}$$

With

$$E_{m'n'}{}^{s'} = -\left[h_{m'}{}^{s'}E_{n'} - h_{m'n'}E^{s'}\right], \quad E_{n'} = \{\alpha E_1,0,0,-i\alpha vE_1\} \tag{3.8}$$

one obtains

$$'E_{m'n'}{}^{s'} = -\left[h_{m'}{}^{s'}\,'E_{n'} - h_{m'n'}\,'E^{s'}\right], \quad 'E_{n'} = \left\{0,0,0,\frac{i}{\rho}\right\} \ . \tag{3.9}$$

The inhomogeneous 3-rank-transformation law of the connexion coefficients (3.3) reduces to a vector addition law of the forces

$$'E_{n'} = E_{n'} + L_{n'} \ . \tag{3.10}$$

Evidently, the force of gravity is dynamically nullified by the Lorentz transformation. No force of gravity is felt by the freely falling observers. The fourth components in (3.5) and (3.9) are attributed to the lateral and radial tidal forces of the freely falling system. We interpret the freely falling observer field

$$'u_{m'} = \{0,0,0,1\}$$

as normal vectors of a shrinking surface comoving with these observers. Then one can define the second fundamental forms of this surface by

$$Q_{\alpha'\beta'} = 'u_{a'||\beta'} = -A_{\beta'\alpha'}{}^{4'}, \quad \alpha',\beta' = 1,2,3 \ . \tag{3.11}$$

Thus, the connexion coefficients split into a spacelike part *A and components containing the second fundamental forms, the tidal field strengths

$$'A_{m'n'}{}^{s'} = {}^{*}A_{m'n'}{}^{s'} + Q_{m'}{}^{s'}\,'u_{n'} - Q_{m'n'}\,'u^{s'} \tag{3.12}$$

with

$$Q_{1'1'} = -'E_{4'} = -\frac{i}{\rho}, \quad Q_{2'2'} = B_{4'} = -\frac{iv}{r}, \quad Q_{3'3'} = C_{4'} = -\frac{iv}{r} \ . \tag{3.13}$$

To get the relation between the two kinds of tidal forces, we use the graded derivatives for the freely falling system including the Lorentz term

$$\Phi_{m'\underset{1}{\|}n'} = \Phi_{m'|n'} - L_{n'm'}{}^{s'}\Phi_{s'}, \qquad \Phi_{m'\underset{2}{\|}n'} = \Phi_{m'|n'} - L_{n'm'}{}^{s'}\Phi_{s'}$$
$$\Phi_{m'\underset{3}{\|}n'} = \Phi_{m'|n'} - \left(B_{n'm'}{}^{s'} + L_{n'm'}{}^{s'}\right)\Phi_{s'} \qquad . \tag{3.14}$$
$$\Phi_{m'\underset{4}{\|}n'} = \Phi_{m'|n'} - \left(B_{n'm'}{}^{s'} + C_{n'm'}{}^{s'} + L_{n'm'}{}^{s'}\right)\Phi_{s'}$$

Dropping from here and throughout the primes on the indices we can write the Ricci of the freely falling system in the auxiliary form

$$\begin{aligned} 'R_{mn} = h_{mn}\left['E^{s}{}_{\underset{1}{\|}s} - 'E^{s}\,'E_{s}\right] - \left[B_{n\underset{2}{\|}m} + B_{n}B_{m}\right] - b_{m}b_{n}\left[B^{s}{}_{\underset{2}{\|}s} + B^{s}B_{s}\right] \\ - \left[C_{n\underset{3}{\|}m} + C_{n}C_{m}\right] - c_{m}c_{n}\left[C^{s}{}_{\underset{3}{\|}s} + C^{s}C_{s}\right] \end{aligned} \tag{3.15}$$

which we analyze with (3.5) and (3.9). In addition, having defined the auxiliary quantities

$$\tilde{Q}_{11} = Q_{11}, \quad \tilde{Q}_{22} = Q_{22}, \quad \tilde{Q}_{33} = Q_{33}, \quad \tilde{Q}_{44} = Q_{11} \ , \tag{3.16}$$

we can establish the expected relation between the two kinds of tidal field strengths with the help of

$$'R_{mn} = -2\tilde{Q}_{m[n}\tilde{Q}_{s]}{}^{s} + h_{mn}\frac{3}{\rho^{2}} \ . \tag{3.17}$$

Using the definitions of the quantities (2.7) and (3.13) one finds

$$A_{mn}A_{rs} = -\tilde{Q}_{mn}\tilde{Q}_{rs} \ . \tag{3.18}$$

If we regard the invariance of the Ricci

$$R_{m'n'} = L^{m\ n}_{m'n'}R_{mn} \tag{3.19}$$

and the form invariance of the right sides of Eqs. (2.9) under Lorentz transformations then it follows from (2.9) and (3.17) that the tidal field strengths get exchanged by the transition from the static system to the freely falling system. Since the tidal field strengths are responsible for the radial and lateral stresses experienced by an extended body and since the stresses are of tensorial form we have decided for a 2-rank representation of the tidal field strengths. In a former paper [14] we have worked out from the Ricci the field equations

$$\begin{aligned} &\left[Q_{mn\wedge s}\,'u^{s} + Q_{mn}Q_{s}{}^{s}\right] = 0 \\ &\left[Q_{s}{}^{s}{}_{\wedge \underline{m}} - Q_{m}{}^{s}{}_{\wedge s}\right] = 0 \qquad . \\ &\left[Q_{s}{}^{s}{}_{\wedge m}\,'u^{m} + Q_{rs}Q^{rs}\right] = 0 \end{aligned} \tag{3.20}$$

The hat denotes the 3-dimensional covariant derivative with respect to the space-like components *A of the connexion coefficients[1]. These equations show the field mechanism and the geometrical background of the forces acting on a freely falling observer.

4. SUMMARY

We have shown that the tidal forces acting on a static observer in the Schwarzschild field are contained in the Einstein field equations which becomes clearly visible if one decomposes these equations into their curvature equations. A Lorentz transformation to a freely falling system substitutes these tidal forces with the tidal forces acting on the freely falling system. In both cases the tidal field strengths can be interpreted as the second fundamental forms of a surface.

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[1] $\underline{m} = 1,2,3$