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REMARKS ON BOLTZMANN'S FORMULATION OF THE HELMHOLTZ VORTICITY THEOREM

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After the death of Treder diskettes with unpublished papers were passed to me. Although the original manuscripts could not be found and therefore a proofreading was not possible, I have nevertheless decided to publish these papers. R. Burghardt

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A dual form of the general relativistic and covariant generalization of the first vorticity theorem of Helmholtz is proven. This dual form is the generalization of Boltzmann's formulation of the Helmholtz theorem.

Schröder and Treder derived (1999, in the following ST1) a general-covariant and general-relativistic kinematic identity was given using tensor calculus from which the general-covariant and general-relativistic form of the first vorticity theorem of Helmholtz can be directly seen and also the form of this vorticity theorem in arbitrary reference systems. Using the symbols of tensor calculus (Ricci-calculus) as defined in ST1, an identity was found between four-velocity

$$u^\alpha = \frac{dx^\alpha}{d\tau}, \quad (\alpha = 0,1,2,3)$$

and its curl:

$$\omega_{\alpha\beta} = u_{\alpha;\beta} - u_{\beta;\alpha} = u_{\alpha,\beta} - u_{\beta,\alpha} \tag{1}$$

This identity is (Eq. 10 in ST1):

$$\omega_{\alpha\beta;\lambda} u^\lambda = \frac{D}{D\tau}\omega_{\alpha\beta} = \omega_\alpha{}^\lambda u_{\lambda;\beta} - \omega_\beta{}^\lambda u_{\lambda:\alpha} + P_{\alpha,\beta} - P_{\beta,\alpha}\,, \tag{2}$$

where

$$P^\alpha = \frac{D^2x^\alpha}{D\tau^2} = \frac{Du^\alpha}{D\tau}$$

is the four-vector of the force referred to the unit rest mass. If the curl disappears:

$$P_{\alpha;\beta} - P_{\beta;\alpha} = P_{\alpha,\beta} - P_{\beta,\alpha} = 0$$

and Eq. (2) is transformed into the general-relativistic form of the Helmholtz vorticity theorem

$$\frac{D}{D\tau}\omega_{\alpha\beta} = \omega_{\alpha}{}^{\lambda}u_{\lambda;\beta} - \omega_{\beta}{}^{\lambda}u_{\lambda;\alpha} \; . \tag{3}$$

In the non-relativistic limiting case of low velocities $\left(v^2 << c^2\right)$ and by choosing inertial frames of reference, this four dimensional equation yields a three-dimensional equation (i,k=1,2,3*)*:

$$\frac{d}{dt}\omega_{ik} = \omega_i{}^l u_{l,k} - \omega_k{}^l u_{l,i} \, . \tag{4}$$

The axial vector ω^i is then introduced into Eq. (4) (ST1, Eq. 19) according to:

$$\omega^i = \frac{1}{2}\varepsilon^{ikl}\omega_{k,l} \tag{5}$$

then Eq. (4) gets the known form of the first Helmholtz vorticity theorem:

$$\frac{d}{dt}\omega^i = \omega^l u^i{}_{,l} \; . \tag{6}$$

The Helmholtz vorticity theorem is also known, in addition to the form in Eq. 6, in the form given by Boltzmann:

$$\frac{d}{dt}\omega_i = \omega^l u_{l,i} \tag{7}$$

Equation (7) follows from Eq. (6) using the identity given by Ertel (1962, 1963a, b, Schröder, 1991):

$$\omega^i\omega_{ik} = 0 \tag{8}$$

The general-covariant relativistic form of this identity is:

$$\omega_{\alpha}{}^{\lambda}u_{\lambda;\beta} - \omega_{\beta}{}^{\lambda}u_{\lambda;\alpha} = \omega_{\alpha}{}^{\lambda}u_{\beta;\lambda} - \omega_{\beta}{}^{\lambda}u_{\alpha;\lambda} \tag{9}$$

(see ST1, Eq. 9a). It is valid with Eq.(1) according to the definition:

$$\omega_{\alpha}{}^{\lambda}u_{\lambda;\beta} = u^{\lambda}{}_{;\alpha}u_{\lambda;\beta} - u_{\alpha;\lambda}u^{\lambda}{}_{;\beta} \tag{10}$$

and

$$\omega_{\alpha}{}^{\lambda}u_{\beta;\lambda} = u^{\lambda}{}_{;\alpha}u_{\beta;\alpha} - u^{\lambda}{}_{;\alpha}u_{\beta;\lambda} \tag{11}$$

and Eq. (9) follows according to the antisymmetrization concerning the indices α and β.

On the basis of Eq. (9), Eq. (3) can also be written in the dual form:

$$\frac{D}{D\tau}\omega_{\alpha\beta} = \omega_{\alpha}{}^{\lambda}u_{\beta;\lambda} - \omega_{\beta}{}^{\lambda}u_{\alpha;\lambda} \tag{12}$$

This dual character of Eqs (3) and (12) means that these equations substitute each other if the antisymmetric tensor $\omega_{\alpha\beta}$ is substituted by the dual tensorial density:

$$\omega^{*uv} = \frac{1}{2}\varepsilon^{uv\alpha\beta}\omega_{\alpha\beta} \tag{13}$$

(using the Levi-Civita symbol $\varepsilon^{uv\alpha\beta}$ for the completely antisymmetric pseudo-tensor). In the non-relativistic limiting case and for inertial frames of reference it follows from Eq. (12) the three-dimensional equation:

$$\frac{d}{dt}\omega_{ik} = \omega_{i}{}^{l}u_{k,l} - \omega_{k}{}^{l}u_{i,l} \tag{14}$$

Using the vorticity vector ω^{i} this Eq. (14) becomes:

$$\frac{d\omega^{i}}{dt} = \omega^{l}u_{l,k}\delta^{ki}$$

i.e. the Boltzmann form of the Helmholtz theorem.

References

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