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THE CURVATURE PARAMETERS IN GRAVITATIONAL MODELS

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Contents

Abstract: The meaning of the curvature parameter is subjected to a rigorous examination concerning metrics of some gravity models. The radial part of the line elements must be brought into the canonical form. It turns out that models are not flat, although the curvature parameter has the value $k = 0$ in comoving coordinates.

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1. INTRODUCTION

For cosmological models, particularly for FRW models, the curvature parameter k was introduced for the classification of the geometry. k is situated in the radial part of the metric if the metric is expressed in comoving coordinates. The typical form of such a metric is

$$ds^2 = \frac{1}{1-k\frac{r'^2}{\mathcal{R}^2}}dr'^2 + r^2 d\Omega^2 - dt'^2.$$

We will call it the canonical form.

For $k=1$ the geometry is called to be positively curved and the universe closed, for $k=0$ flat and open, for $k=-1$ negatively curved and open. This terminology has been transferred to collapsing gravity models by McVittie [1] because the collapse of a stellar object is mathematically closely related to the expansion of the universe. However, we have reasonable doubts that the curvature parameter could represent the curvature of a geometric model. We investigate for several models which interpretation might have the quantity k. The canonical form of the metric plays an essential role. In the following sections, we will examine some models considering their spatial curvature properties.

2. EXTERIOR SCHWARZSCHILD SOLUTION, STATIC

The Schwarzschild line element in the standard form is

$$ds^2 = \frac{1}{1-\frac{2M}{r}}dr^2 + r^2 d\Omega^2 - \left(1-\frac{2M}{r}\right)dt^2 \ . \qquad (2.1)$$

It is to be brought into the canonical form. For this we need the radius of curvature of the Schwarzschild parabola and an associated quantity $\mathcal{R}$

$$\rho = \sqrt{\frac{2r^3}{M}} = 2\mathcal{R}, \quad \mathcal{R} = \sqrt{\frac{r^3}{2M}} \ . \qquad (2.2)$$

The new quantity $\mathcal{R}$ can be geometrically interpreted. If one extends the curvature vector of the Schwarzschild parabola to its directrix, the distance $\mathcal{R}$ is excised between the directrix and the Schwarzschild parabola.

The speed of an object which is in free fall in the Schwarzschild field can be rewritten with (2.2) in the form

$$v = -\sqrt{\frac{2M}{r}} = -\frac{r}{\mathcal{R}}, \qquad (2.3)$$

whereby

$$r = \mathcal{R}\sin\eta \qquad (2.4)$$

applies. Thus, the Schwarzschild metric can also be brought into the form of de Sitter

$$ds^2 = \frac{1}{\cos^2\eta} dr^2 + r^2 d\Omega^2 - \cos^2\eta\, dt^2 \quad (2.5)$$

or into the canonical form

$$ds^2 = \frac{1}{1-\frac{r^2}{\mathcal{R}^2}} dr^2 + r^2 d\Omega^2 - \left(1-\frac{r^2}{\mathcal{R}^2}\right) dt^2 . \quad (2.6)$$

The spatial part of this metric might suggest that it describes a spherical geometry. However, it must be remembered that $\mathcal{R}$ is a function $\mathcal{R} = \mathcal{R}(r)$ and also that the origin of $\mathcal{R}$ is not fixed but moves on the directrix of the Schwarzschild parabola. The spatial part of (2.6) still describes Flamm's paraboloid.

If the radial arc element is generally written in the form

$$\left(dx^1\right)^2 = \frac{1}{1-k\frac{r^2}{\mathcal{R}^2}} dr^2 , \quad (2.7)$$

it can be seen that for the Schwarzschild geometry the quantity k takes the value

$$k = 1. \quad (2.8)$$

Flamm's paraboloid correctly appears *positively* curved and as can seen from the canonical representation (2.6) of the metric is *open.* That $k = 1$ in this case describes an open model, extending to the infinite, is not covered by the FRW definition of k. We no longer want to associate the meaning of k with the curvature of the geometry, but with the shape of the metric. We will call k as *form parameter* of the model.

3. EXTERIOR SCHWARZSCHILD SOLUTION, FREE FALL FROM INFINITY

A coordinate transformation of the Schwarzschild metric, which has been derived by Lemaître [2], will be treated in relation to the problems addressed by us. One can simplify the substitution of Lemaître

$$r = \sqrt[3]{2M}\left[\frac{3}{2}\left(r'' - t''\right)\right]^{\frac{2}{3}}, \quad t'' = t + 2\sqrt{2Mr} + 2M \ln\frac{1-\sqrt{\frac{2M}{r}}}{1+\sqrt{\frac{2M}{r}}} \quad (3.1)$$

with

$$\mathcal{R}(r'', t'') = \frac{3}{2}\left(r'' - t''\right)$$

to

$$r^3 = 2M\mathcal{R}^2 .$$

From the Schwarzschild definition $v=-\sqrt{2M/r}$ one obtains for the velocity of observers falling from the infinite the expression

$$v=-\frac{r}{\mathcal{R}}=-\sqrt[3]{\frac{2M}{\mathcal{R}}} \tag{3.2}$$

and one has recovered the relations (2.2), (2.3) with the help of the Lemaître transformation.

Differentiating (3.1) and substituting into the standard form of the Schwarzschild metric, one obtains the Lemaître metric

$$ds^2=\mathcal{K}^2\left[dr''^2+\mathcal{R}^2 d\Omega^2\right]-dt''^2,\quad \mathcal{K}=-v\left(r'',t''\right)=\frac{r}{\mathcal{R}}. \tag{3.3}$$

$\mathcal{K}$ is called scale factor. For the metric that describes a freely falling observer one has

$$k=0. \tag{3.4}$$

Without doubt, the coordinate transformation (3.1) cannot change the geometric structure of the model. The falling object moves on a radial curve of Flamm's paraboloid, which is positively curved. The Lemaître coordinate system follows the falling observer and shrinks toward the event horizon.

From the line element (3.3) we see that the time-like metric factor is $g_{44}=1$, ie that the time-like arc element is flat. For comoving observers coming in free fall from infinity, applies the same time at any position $r>2M$. The proper time of the observer is identical with the time coordinate t'' and agrees with the coordinate time of an observer at rest at infinity. A constant timelike metric factor has as a consequence that no gravity acts on the observer. This corresponds to the principle of Einstein's elevator.

From the metric (3.3) we calculate the lateral field quantities with the tetrad technology. We refer the interested reader to our presentations in [3]. For the spatial parts of these quantities one has

$$B_{\alpha'}=\left\{\frac{1}{r},0,0\right\},\quad C_{\alpha'}=\left\{\frac{1}{r},\frac{1}{r}\cot\vartheta,0\right\},\quad \alpha'=1',2',3'. \tag{3.5}$$

In contrast, in the static case one has

$$B_{\alpha}=\left\{\frac{a}{r},0,0\right\},\quad C_{\alpha}=\left\{\frac{a}{r},\frac{1}{r}\cot\vartheta,0\right\},\quad a=\sqrt{1-2M/r}=\sqrt{1-r^2/\mathcal{R}^2}. \tag{3.6}$$

While (3.5) are the expressions for a flat geometry, (3.6) is typical of a curved geometry. The form of (3.3) and (3.5) seems to indicate that $k=0$ stands for a flat geometry. We want to get to the bottom of the matter.

If one establishes the matrix $\Lambda^{i'}_{i}=x^{i'}{}_{|i}$ by differentiation of (3.1) and further reads the 4-beine from (2.1) and (3.3), one obtains with $L^{m'}_{m}=\underset{i'}{\overset{m'}{e}}\Lambda^{i'}_{i}\underset{m}{e}{}^{i}$ the matrix of the Lorentz transformation

$$L_{m}^{m'} = \begin{pmatrix} \alpha & & & i\alpha v \\ & 1 & & \\ & & 1 & \\ -i\alpha v & & & \alpha \end{pmatrix}, \quad \alpha = \frac{1}{\sqrt{1-\frac{2M}{r}}}, \quad v = -\sqrt{\frac{2M}{r}}, \tag{3.7}$$

which describes the transition from the static reference system to the freely falling reference system. With $B_{m'} = L_{m'}^{m} B_m$ one obtains

$$B_{\alpha'} = \left\{\alpha a \frac{1}{r}, 0, 0\right\} = \left\{\frac{1}{r}, 0, 0\right\}. \tag{3.8}$$

In this expression the curvature quantity a is still included. It is hidden by the kinematic quantity α according to $\alpha a = 1$. It is clear that $k = 0$ cannot be a condition for a flat geometry. It is therefore quite justified if we deny k the property to make statements about the curvature of a geometry. The quantity k classifies the structure of a metric and deserves to be called form parameter of the metric.

4. FREE FALL FROM AN ARBITRARY POSITION

The problem treated in the last section can be generalized to the free fall from an arbitrary position in the Schwarzschild field. To find the transition to a coordinate system that accompanies a freely falling object, one needs the speed v' of this object. It is not known a priori, but can be determined circuitously, by starting from the known velocity v of an observer who comes from infinity. In addition, one faces the velocity $v_0 = \text{const.}$ of an observer who is falling from infinity as well. This velocity is measured when the observer has reached the position r_0 namely, that position from which the first observer is emittet. The two velocities v and v_0 are subtracted relativistically. According to Einsein's law of addition of velocities one has

$$v' = v'(r, r_0) = \frac{v - v_0}{1 - v v_o} = \frac{-\sqrt{\frac{2M}{r}} - \left(-\sqrt{\frac{2M}{r_0}}\right)}{1 - \sqrt{\frac{2M}{r}}\sqrt{\frac{2M}{r_0}}}. \tag{4.1}$$

In a paper [5] we have studied the problem in more detail and have found the transformation matrix

$$\begin{matrix} \Lambda_{1'}^{1} = -\frac{\alpha'^2}{\alpha^2} v', & \Lambda_{4'}^{1} = -i\frac{\alpha'^2}{\alpha^2} v', & \Lambda_{1'}^{4} = -i\alpha'^2 v'^2, & \Lambda_{4'}^{4} = \alpha'^2 \\ \Lambda_{1}^{1'} = -\frac{\alpha^2}{v'}, & \Lambda_{4}^{1'} = -i, & \Lambda_{1}^{4'} = -i\alpha^2 v', & \Lambda_{4}^{4'} = 1 \end{matrix}. \tag{4.2}$$

It leads to the line element

$$ds^2 = \alpha_0^2 (v - v_0)^2 dr'^2 + r^2 d\Omega^2 - \alpha_0^2 (1 - v v_0)^2 dt'^2 \tag{4.3}$$

from which one can read $k = 0$ as well. Little surprisingly, we get the same result as for an observer who comes from infinity.

5. THE R_h = ct MODEL

Melia [9] has proposed a flat model which explains better the observed phenomena in the cosmos than the standard model. The cosmic horizon is at that distance from any observer, in which the recession velocities of the galaxies have the velocity of light. The cosmos expands linearly, thus it does not accelerate.

Recently we have proposed a cosmological model [10] which we have called pressure model. It is based on an exact solution of Einstein's field equations and includes pressure and mass density. The model is based on a pseudo-hypersphere, the spatial curvature is positive $(k = 1)$. It is noteworthy that this model leads to the same results as Melias model, ie to a model with linear unaccelerated expansion.

Thus, the question arises whether that concordance is accidental, or whether the models are identical despite the inconsistencies regarding the curvature of space. We have investigated this question.

Melia primarily realizes his concept in the comoving system. The line element for $k = 0$ has the form

$$ds^2 = \mathcal{K}^2\left[dr'^2 + r'^2 d\Omega^2\right] - dt'^2, \tag{5.1}$$

whereby the scale factor is proportional to the cosmic time t' according to the linear expansion of the universe. Therefore we put

$$\mathcal{K} = \frac{t'}{\mathcal{R}_0} \tag{5.2}$$

with $\mathcal{R}_0$ as constant factor which can be set to 1 without loss of generality. From the above line element we read the 4-bein system

$$\overset{1'}{e}_{1'} = \mathcal{K}, \quad \overset{2'}{e}_{2'} = \mathcal{K} r' = r, \quad \overset{3'}{e}_{2'} = \mathcal{K} r' \sin\vartheta = r\sin\vartheta, \quad \overset{4'}{e}_{4'} = 1 \tag{5.3}$$

and therefrom calculate the components of the Ricci-rotation coefficients

$$'U_{m'} = \left\{0,0,0,-\frac{i}{\mathcal{R}}\right\}, \quad B_{m'} = \left\{\frac{1}{r},0,0,-\frac{i}{\mathcal{R}}\right\}, \quad C_{m'} = \left\{\frac{1}{r},\frac{1}{r}\cot\vartheta,0,-\frac{i}{\mathcal{R}}\right\}. \tag{5.4}$$

However, these quantities are identical with those which we have derived for our pressure model. Herewith the field equations yield the stress-energy tensor with the pressure and the mass density

$$\kappa p = -\frac{1}{\mathcal{R}^2}, \quad \kappa\mu_0 = \frac{3}{\mathcal{R}^2} \tag{5.5}$$

and the equation of state

$$p = -\frac{1}{3}\mu_0 . \tag{5.6}$$

All quantities of Melia are identical with those of our pressure model. Therefore it is reasonable to assume that the model of Melia is identical with ours and thus positively curved and finite. In this case k cannot be interpreted as curvature parameter either. The apparently flat structure of the lateral field quantities in (5.4) can be explained in the same way as in the previous sections. The cosmos is expanding in free fall. Again, the principle of Einstein's elevator applies.

6. THE COSMOLOGICAL MODELS OF THE DE SITTER FAMILY

The dS family consists of four models, the actual dS model, the model of Lanczos, Lanczos-like model and the anti-de Sitter model.

The line element of de Sitter in its static form is

$$ds^2 = \frac{1}{\cos^2\eta}dr^2 + r^2 d\Omega^2 - \cos^2\eta\, dt^2 = \frac{1}{1-\frac{r^2}{\mathcal{R}^2}}dr^2 + r^2 d\Omega^2 - \left(1-\frac{r^2}{\mathcal{R}^2}\right)dt^2 . \tag{6.1}$$

From its canonical form can be read $k = 1$. The dS cosmos is positively curved and therefore closed. It has an embedding. (6.1) is the line element on a pseudo-hyper sphere with a constant radius. From Lemaître stems a coordinate transformation which results in the line element

$$ds^2 = \mathcal{K}^2\left[dr'^2 + r'^2 d\Omega^2\right] - dt'^2, \quad \mathcal{K} = e^{\psi'} \tag{6.2}$$

which can be interpreted as a metric of an expanding universe. Evidently one has $k = 0$ and this is what is stated in the literature. The expanding dS cosmos is flat and open. We do not believe that the geometric structure of a model can be changed by a coordinate transformation. We tend to believe that they are two different universes, which are determined by different slices on the pseudo-hyper sphere with constant radius. In this light the Lemaître transformation is only a model generating mathematical method. The two coordinate systems $\{r', t'\}$ and $\{r, t\}$ belong to different models and cannot be interpreted as systems which comove or which do not comove with the expansion.

But we have to consider that the coordinate transformation of Lemaître is accompanied by a Lorentz transformation, which apparently is often overlooked. It is not entirely clear why a locally operating Lorentz transformation should have global consequences, such as the redeployment of slices of an unmodifiable pseudo-hyper sphere. Therefore, we prefer a different interpretation of the Lemaître transformation. Not space is expanding, but the coordinate system. Observers, associated with this coordinate system are moving from an arbitrary point into every direction. In this case, k cannot be associated with the curvature of space, but with the form parameter of the metric.

The motion occurs in free fall. This has the consequence that different forces act on the comoving and on the non-comoving observers. The relation of the forces can be calculated via the inhomogeneous transformation law of the Ricci-rotation coefficients

$$'U_{m'} = L^{m}_{m'}U_m + 'L_{m'} . \tag{6.3}$$

In the non-comoving system U has only one component, the force in the radial direction.

$$U_m = \{U_1, 0, 0, 0\} . \tag{6.4}$$

However, in the comoving system one has

$$'U_{m'} = \{0, 0, 0, 'U_{4'}\} . \tag{6.5}$$

As required by the principle of Einstein's elevator, no radial forces act on the observer. However, the tidal force $'U_{4'}$ emerges. It is responsible for the expansion of a volume element into the radial direction. The lateral field quantities of dS cosmos take the flat form (3.8) in the expanding system just as they do in the Schwarzschild model. However, for them applies that the geometric quantity $a = \cos\eta$ is repealed by the kinematic quantity $\alpha = 1/\sqrt{1-v^2}$.

The line element of the *Lanczos cosmos* has in the non-comoving case with $r = \mathcal{R}_0 \sin\eta$ the form

$$ds^2 = \frac{1}{\cos^2\eta}dr^2 + r^2 d\Omega^2 - \cos^2\eta dt^2 = \frac{1}{1-\frac{r^2}{\mathcal{R}_0^2}}dr^2 + r^2 d\Omega^2 - \left(1-\frac{r^2}{\mathcal{R}_0^2}\right)dt^2 \tag{6.6}$$

and in non-comoving coordinates with $r' = \mathcal{R}_0 \sin\eta'$

$$ds^2 = \mathcal{K}^2\left[\frac{1}{\cos^2\eta'}dr'^2 + r'^2 d\Omega^2\right] - dt'^2 = \mathcal{K}^2\left[\frac{1}{1-\frac{r'^2}{\mathcal{R}_0^2}}dr'^2 + r'^2 d\Omega^2\right] - dt'^2 . \tag{6.7}$$

Therein $\mathcal{K} = \cos i\psi' = \mathrm{ch}\psi'$ is the scale factor and on the basis of $t' = \mathcal{R}_0\psi'$ a time-dependent quantity and $\mathcal{R}_0$ a constant. With $\mathcal{R} = \mathcal{K}\mathcal{R}_0$ can (6.7) be written in the form

$$ds^2 = \mathcal{R}^2 d\eta'^2 + \mathcal{R}^2 \sin^2\eta' d\Omega^2 + \mathcal{R}_0^2\, di\psi'^2 . \tag{6.8}$$

It stands for a pseudo-hyper-sphere with time-dependent radius $\mathcal{R}$. From the canonical form of the metric one reads $k = 1$ for both the static form and for the expanding form of the model as well. Here the meaning of k would be justified as curvature parameter, but also alike as the form parameter of the metric.

For the Lanczos-like cosmos applies in the non-comoving case the line element

$$ds^2 = \frac{1}{1-\mathrm{sh}^2\eta}dr^2 + r^2 d\Omega^2 - \left(1-\mathrm{sh}^2\eta\right)dt^2 = \frac{1}{1-\frac{r^2}{\mathcal{R}_0^2}}dr^2 + r^2 d\Omega^2 - \left(1-\frac{r^2}{\mathcal{R}_0^2}\right)dt^2, \qquad (6.9)$$

and for the comoving observer

$$ds^2 = \mathcal{K}^2\left[\frac{1}{1+\mathrm{sh}^2\eta'}dr'^2 + r'^2 d\Omega^2\right] - dt'^2 = \mathcal{K}^2\left[\frac{1}{1+\frac{r'^2}{\mathcal{R}_0^2}}dr'^2 + r'^2 d\Omega^2\right] - dt'^2. \qquad (6.10)$$

Remarkably, the universe is for the static observers positively curved and finite $(k=1)$. For the observer who participates in the expansion negatively curved and open $(k=-1)$. The first observer lives in a universe with a finite number of stars, the second observer is in an infinite universe with infinitely many stars. The Lanczos-like universe is contradictory and cannot be the basis for a physically suitable model.

The *anti-de Sitter cosmos* has in non-comoving coordinates the line element

$$ds^2 = \frac{1}{\mathrm{ch}^2\eta}dr^2 + r^2 d\Omega^2 - \mathrm{ch}^2\eta\, dt^2 = \frac{1}{1+\frac{r^2}{\mathcal{R}_0^2}}dr^2 + r^2 d\Omega^2 - \left(1+\frac{r^2}{\mathcal{R}_0^2}\right)dt^2 \qquad (6.11)$$

and in comoving coordinates

$$ds^2 = \mathcal{K}^2\left[\frac{1}{\mathrm{ch}\eta'^2}dr'^2 + r'^2 d\Omega^2\right] - dt'^2 = \mathcal{K}^2\left[\frac{1}{1+\frac{r'^2}{\mathcal{R}_0^2}}dr'^2 + r'^2 d\Omega^2\right] - dt'^2. \qquad (6.12)$$

In both cases is $(k=-1)$. The universe is negatively curved for all observers and open.

7. COLLAPSING MODELS

The classic collapsing gravity models include the model of Oppenheimer and Snyder [6] and the models of McVittie [1]. McVittie has undertaken the FRW classification for the curvature of models from cosmology to gravitational theory. Thus, the model $k=0$ is ident with the one of OS, $k=1$ was later rediscovered by Weinberg [7] and discussed by him in detail.

The model of OS describes a star which initially filled the infinite universe and had vanishing mass density. The star collapses in free fall, falls through the event horizon and shrinks to a point with infinitely high mass density and space curvature.

The OS-metric in comoving coordinates is

$$ds^2 = \mathcal{K}^2\left[dr'^2 + r'^2 d\Omega^2\right] - dt'^2 \tag{7.1}$$

with

$$r = \mathcal{K} r', \quad \mathcal{K} = \left(1 - \frac{3}{\rho'_g} t'\right)^{\frac{2}{3}}, \quad \rho'_g = \sqrt{\frac{2r'_g}{M}} \ . \tag{7.2}$$

Therein r'_g is the comoving coordinate on the surface of the star, ie, at the boundary surface between of the interior OS solution and the exterior Schwarzschild solution. The OS-metric in non-comoving coordinates has the form

$$ds^2 = \frac{1}{1 - \frac{r^2}{\mathcal{R}_g^2}} dr^2 + r^2 d\Omega^2 + g_{44} dit^2 , \tag{7.3}$$

whereby for g_{44} is given a somewhat complicated expression by OS. (7.1) is of type $k = 0$ and (7.3) of type $k = 1$. The stellar object of the OS model collapses in free fall. The geometry of the comoving observer *appears* to him flat, but is a spherical cap according to (7.3) and thus positively curved. It should be noted that in the OS model the metric of the stellar object merges into the exterior Schwarzschild solution at the boundary surface between the inner and outer regions, but the first derivatives of the metrics do not match.

In the model of McVittie-Weinberg the star does not collapse from the infinite but from a finite position. The metric in comoving coordinates is

$$ds^2 = \mathcal{K}^2\left[\frac{1}{1 - \frac{r'^2}{\mathcal{R}_0^2}} dr'^2 + r'^2 d\Omega^2\right] - dt'^2 \tag{7.4}$$

and in non-comoving coordinates

$$ds^2 = \frac{1}{1 - \frac{r^2}{\mathcal{R}^2}} dr^2 + r^2 d\Omega^2 + g_{44} dit^2 . \tag{7.5}$$

For g_{44} Weinberg also gives a complicated expression but closer working through the theory leads to contradictions. From (7.4) one can read $g_{4'4'} = 1$. However, such an approach is only valid for a collapse from the infinite and not for a collapse from an arbitrary position. Both line elements have canonical forms. From them one takes $k = 1$. The 3-dimensional space is positively curved. It is a spherical cap.

In [8] we have extended the interior Schwarzschild solution to a collapsing model. A cap of a sphere with $k = 1$ slides down Flamm's paraboloid. Since we have largely omitted the coordinate method, the k-problem does not occur in this model.

8. CONCLUSIONS

We have examined several gravity models, static ones with freely falling observers, collapsing and expanding ones with respect to their curvature properties. We have found that the FRW classification by means of the quantity k interpreted as curvature parameter is not reliable. In particular, $k=0$ does not mean that the spatial part of the model is flat. $k=0$ rather suggests that a motion takes place in free fall and that freely falling observers are not subjected to forces. In particular, we believe that the $R_h = ct$ model of Melia [9, 10], which claims to be $k=0$, is not flat, but is positively curved. We have obtained results, which we present in a table:

model	comoving	non-comoving	geometry
dS	$k=0$	$k=1$	sphere $\mathcal{R} = \text{const.}$
Lanczos	$k=1$	$k=1$	sphere $\mathcal{R} = \text{const.}$
Lanczos-like	$k=-1$	$k=-1$	pseudo-sphere $\mathcal{R} = \text{const.}$
AdS	$k=-1$	$k=-1$	pseudo-sphere $\mathcal{R} = \text{const.}$
Melia	$k=0$	--	flat
Burghardt	$k=0$	$k=1$	sphere $\mathcal{R} = \mathcal{R}(t)$
Friedman	$k=1,0,-1$	--	sphere, flat, pseudo-sphere
OS	$k=0$	$k=1$	cap of a sphere
Weinberg	$k=1$	$k=1$	cap of a sphere
SS	$k=0$	$k=1$	paraboloid

9. LITERATURE

[1] McVittie G. C., *Gravitational collapse to a small volume.* Astrophys. J. **140**, 401, 1964

[2] Lemaître G., *L'univers en expansion*. Ann. Sci. Brux. **A 53**, 51, 1933

[3] Burghardt R. *Spacetime curvature*. http://members.wavenet.at/arg/EMono.htm

[4] Burghardt R., *Transformations in de Sitter and Lanczos models I.* . arg.or.at, Report ARG-2016-03

[5] Burghardt R. *Einstein's elevator*. arg.or.at, Report ARG-2011-02

[6] Oppenheimer J. R., Snyder H., *On continued gravitational contraction.* Phys. Rev. **56**, 455, 1939

[7] Weinberg S., *Gravitation and Cosmology*. John Wiley & Sons, New York 1997

[8] Burghardt R., *Collapsing Schwarzschild interior soluition.* Journ. Mod. Phys. **6**, 1895, 2015 http://dx.doi.org/10.4236/jmp.2015.613195

[9] Melia F., Abdelqader M., *The cosmological spacetime.* astro-ph/09075394

[10] Melia F., Shevchuk A. S. H., *The* R_h*=ct universe.* Mon. Not. Roy. Astr. Soc. **419**, 2579, 2012