ARG Austrian Reports on Gravitation

ARG-2007-01

REMARKS ON THE DE SITTER MODEL

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Keywords: de Sitter, embedding, second fundamental forms, expansion, cosmological constant

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The de Sitter model is reinvestigated by means of the theory of surfaces. It is shown that the cosmological constant has a natural geometrical explanation. The transition from the static version of the model to the expanding version is performed with a Lorentz transformation.

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1. INTRODUCTION

We show the different ways one could interpret curvature of space and we point out the consequences for gravitation theory by applying these different interpretations of curvature.

We face the following different views how to understand curvature:

(I) Spaces have curvature, if they are non-Euclidean. The space is not curved, but the geometry, the curvature is determined by the metric. A higher dimensional flat space for embedding the four-dimensional world does not exist. This point of view has the advantage that one can treat solutions of the Einstein field equations also for cases where surfaces do not exist.

(II) The geometry might be described by embedding. However, this is not necessary as the curvature can be expressed by the intrinsic properties of the 4-dimensional space.

(III) The geometry is explained by embedding surfaces into a higher dimensional flat space. The main advantage of this method is that one can utilize the tools of differential geometry like Gauss and Codazzi equations, which could give some insight into the geometrical structure of the model.

2. THE STATIC VERSION

In a flat 5-dimesional space we introduce a Cartesian co-ordinate system, which is related to a spherical co-ordinate system by

$$\begin{aligned} x^{3'} &= \mathfrak{R} \sin\eta \sin\vartheta \sin\varphi \\ x^{2'} &= \mathfrak{R} \sin\eta \sin\vartheta \cos\varphi \\ x^{1'} &= \mathfrak{R} \sin\eta \cos\vartheta \qquad , \\ x^{4'} &= \mathfrak{R} \cos\eta \sin i\psi \\ x^{0'} &= \mathfrak{R} \cos\eta \cos i\psi \end{aligned} \qquad (2.1)$$

wherein $\mathfrak{R}$ =const. is the curvature radius of the pseudo-hypersphere

$$x^{a'} x^{a'} = \mathfrak{R}^2, \quad a' = 0', \ldots, 4' \qquad (2.2)$$

and

$$\mathbb{R}\, d i\psi = i dt \tag{2.3}$$

is the increase of the arc length of the iψ-lines and is identified with the flow of time. To shed light on the geometrical background of the model we want to emphasize that $x^2 + y^2 = \mathbb{R}^2$, $x^2 - y^2 = \mathbb{R}^2$, $x^2 + (iy)^2 = \mathbb{R}^2$ are three different objects: a circle, a hyperbola, and pseudo-circle, also called a hyperbola of constant curvature. In the last case the imaginary number i is soldered to the variable y. As it is not possible to imagine a complex object, one uses pseudo-real representations for these objects. Although the pseudo-real representations are evidently 'wrong', their use might give some insight into the geometry. Mostly the pseudo-circle is drawn as a hyperbola. Such a figure correctly shows the range of the aperture angle as [∞, +∞] but exhibits a non-constant curvature. However, this figure does not show that all points of the corresponding pseudo-circle are equal. Just this is important for the geometrical interpretation of (2.1), (2.2) and (2.3). We stick to the convention that t is the real number we read from a clock. We need this number for the imaginary entity it. Applied to our cosmological problem the arc $\mathbb{R}\, i\psi$ plays the role of the time. Suppressing two spacelike dimensions the model is drawn as a hyperboloid of revolution in the pseudo-real representation. It has the inconvenient features discussed above and in addition, spacelike circular slices appear as ellipses while they are circles in the complex space.

From (2.1) we deduce the 5-dimensional flat-space metric in pseudo-polar co-ordinates as

$$ds^2 = d\mathbb{R}^2 + \mathbb{R}^2 d\eta^2 + \mathbb{R}^2 \sin^2\eta\, d\vartheta^2 + \mathbb{R}^2 \sin^2\eta \sin^2\vartheta\, d\varphi^2 + \mathbb{R}^2 \cos^2\eta\, di\psi^2 \; . \tag{2.4}$$

For $\mathbb{R}$ = const. we obtain the metric for the pseudo-spherical surface, we are dealing with. From (2.4) we read from the 5-bein

$$\overset{0}{e}_0 = 1, \quad \overset{1}{e}_1 = \mathbb{R}, \quad \overset{2}{e}_2 = \mathbb{R}\sin\eta, \quad \overset{3}{e}_3 = \mathbb{R}\sin\eta\sin\vartheta, \quad \overset{4}{e}_4 = \mathbb{R}\cos\eta \tag{2.5}$$

and calculate the Ricci rotation coefficients as

$$A_{ab}{}^{c} = M_{ab}{}^{c} + B_{ab}{}^{c} + C_{ab}{}^{c} + E_{ab}{}^{c}, \quad a = 0,1,\ldots,4 \; . \tag{2.6}$$

With the help of the unit vectors

$$m_a = \{0,1,0,0,0\}, \quad b_a = \{0,0,1,0,0\}, \quad c_a = \{0,0,0,1,0\}, \quad u_a = \{0,0,0,0,1\} \tag{2.7}$$

we get

$$\begin{aligned} M_{ab}{}^{c} &= m_a M_b m^c - m_a m_b M^c, \quad B_{ab}{}^{c} = b_a B_b b^c - b_a b_b B^c \\ C_{ab}{}^{c} &= c_a C_b c^c - c_a c_b C^c, \quad E_{ab}{}^{c} = -u_a E_b u^c + u_a u_b E^c \end{aligned} \; , \tag{2.8}$$

wherein

$$M_a=\left\{\frac{1}{\mathcal{R}},0,0,0,0\right\},\quad B_a=\left\{\frac{1}{\mathcal{R}},\frac{1}{\mathcal{R}}\cot\eta,0,0,0\right\}$$
$$C_a=\left\{\frac{1}{\mathcal{R}},\frac{1}{\mathcal{R}}\cot\eta,\frac{1}{\mathcal{R}\sin\eta}\cot\vartheta,0,0\right\},\quad E_a=\left\{-\frac{1}{\mathcal{R}},\frac{1}{\mathcal{R}}\tan\eta,0,0,0\right\}. \tag{2.9}$$

$\mathcal{R},\mathcal{R}\sin\eta,\mathcal{R}\sin\eta\sin\vartheta,\mathcal{R}\cos\eta$ are the only non-vanishing components of the normal and odd curvature vectors of the pseudo-sphere in the polar co-ordinate system. The equations for those curvatures are

$$\begin{aligned}
&M_{a\underset{1}{|||}b}+M_aM_b=0,\quad M^c{}_{\underset{1}{|||}c}+M^cM_c=0\\
&B_{a\underset{2}{|||}b}+B_aB_b=0,\quad B^c{}_{\underset{2}{|||}c}+B^cB_c=0\\
&C_{a\underset{3}{|||}b}+C_aC_b=0,\quad C^c{}_{\underset{3}{|||}c}+C^cC_c=0\\
&E_{a\underset{4}{|||}b}-E_aE_b=0,\quad E^c{}_{\underset{4}{|||}c}-E^cE_c=0
\end{aligned}. \tag{2.10}$$

They decouple from the flat-space field equations $R_{ab}\equiv 0$. We have used the graded covariant derivatives introduced in former papers [1] with the properties

$$\begin{aligned}
&m_{a\underset{1}{|||}b}=m_{a|b}=0,\quad b_{a\underset{2}{|||}b}=b_{a|b}-M_{ba}{}^{c}b_c=0\\
&c_{a\underset{3}{|||}b}=c_{a|b}-M_{ba}{}^{c}c_c-B_{ba}{}^{c}c_c=0,\quad u_{a\underset{4}{|||}b}=u_{a|b}-M_{ba}{}^{c}u_c-B_{ba}{}^{c}u_c-C_{ba}{}^{c}u_c=0
\end{aligned}. \tag{2.11}$$

Reduction to four dimensions by shifting all 0-terms to the right side of the field equations and using the 4-dimensional graded derivatives in analogy to (2.11) we obtain

$$\begin{aligned}
R_{mn}=&-\left[B_{n\underset{2}{||}m}+B_nB_m\right]-b_nb_m\left[B^s{}_{\underset{2}{||}s}+B^sB_s\right]\\
&-\left[C_{n\underset{3}{||}m}+C_nC_m\right]-c_nc_m\left[C^s{}_{\underset{3}{||}s}+C^sC_s\right]\\
&+\left[E_{n\underset{4}{||}m}-E_nE_m\right]+u_nu_m\left[E^s{}_{\underset{4}{||}s}-E^sE_s\right]\\
=&\,m_nm_m\left[M_0B_0+M_0C_0-M_0E_0\right]\\
&+b_nb_m\left[B_0M_0+B_0C_0-B_0E_0\right]\\
&+c_nc_m\left[C_0M_0+C_0B_0-C_0E_0\right]\\
&-u_nu_m\left[E_0M_0+E_0B_0+E_0C_0\right]
\end{aligned}. \tag{2.12}$$

Inserting the values (2.9) into (2.12), we obtain the field equations of the de Sitter universe [2, 3]

$$R_{mn} - \frac{1}{2} R g_{mn} + \lambda g_{mn} = 0, \quad \lambda = \frac{3}{\mathcal{R}^2} \ . \tag{2.13}$$

We realize that the cosmological constant λ has a geometrical explanation and we will work this out in more detail. Redefining

$$M_0 = A_{11}, \quad B_0 = A_{22}, \quad C_0 = A_{33}, \quad E_0 = -A_{44} \tag{2.14}$$

as second fundamental forms of the surface, we can split the connexion coefficients into

$$A_{ab}{}^{c} = {}'A_{ab}{}^{c} + n_b A_a{}^{c} - n^c A_{ab} \ , \tag{2.15}$$

wherein 'A is the 4-dimensional part of the connexion coefficients and $n_a = \{1,0,0,0,0\}$ is the normal vector of the surface. Separating the 4-dimensional part 'R of the Ricci by

$$R_{ab} = {}'R_{ab} + 2A_{a[c}A_{b]}{}^{c} \equiv 0, \quad R = {}'R + 2A^{d}{}_{[c}A_{d]}{}^{c} \ , \tag{2.16}$$

we get with

$$2A_{m[s}A_{n]}{}^{s} - g_{mn}A_{[r}{}^{s}A_{s]}{}^{r} = \lambda g_{mn} \tag{2.17}$$

the field equations (2.13). This shows that the cosmological constant can be derived from embedding and the de Sitter field equations result from the Gauss equations. In the field equations (2.12), (2.13) a force E_m appears, pointing into the 1-direction. As all points of the pseudo-hyper sphere are equal and the 1-direction on the spacelike sphere is arbitrary this force can point into any direction. We assume that the static version is not a proper representation of the de Sitter model, but we think that this force may be related to an expansion of the universe.

3. THE EXPANDING VERSION

Lemaître [4] and Robertson [5] have shown that a change of co-ordinates

$$r = \mathcal{R}\sin\eta, \quad \bar{r} = re^{-\bar{\psi}}, \quad \bar{\psi} = \psi + \ln\cos\eta, \quad \bar{t} = \mathcal{R}\bar{\psi} \tag{3.1}$$

transforms the metric to

$$ds^2 = e^{2\bar{\psi}}\left[d\bar{r}^2 + \bar{r}^2 d\vartheta^2 + \bar{r}^2 \sin^2\vartheta d\varphi^2\right] - d\bar{t}^2 . \tag{3.2}$$

The new form of the metric appears to be flat and the spacelike part to be time-dependent. We differentiate (3.1) and we multiply these differentials with the components of the 4-bein, which we read from

$$dx^{1'} = e^{\bar{\psi}} d\bar{r}, \quad dx^{4'} = \mathcal{R} d i\bar{\psi} . \tag{3.3}$$

Thus, we obtain the Lorentz transformation

$$dx^{m'} = L_m^{m'} dx^m, \quad L_1^{1'} = \cos i\chi, \quad L_4^{1'} = \sin i\chi, \quad L_1^{4'} = -\sin i\chi, \quad L_4^{4'} = \cos i\chi \tag{3.4}$$

with the definitions

$$\cos i\chi = \alpha, \quad \sin i\chi = i\alpha v . \tag{3.5}$$

The new moving observers have the velocity

$$v = \sin\eta = \frac{r}{\mathcal{R}} \tag{3.6}$$

and

$$\alpha = \frac{1}{\cos\eta} = \frac{1}{\sqrt{1 - \frac{r^2}{\mathcal{R}^2}}} \tag{3.7}$$

is the Lorentz factor of this transformation. (3.6) describes the velocity of expansion of the universe and is related to the Hubble constant. Performing the Lorentz transformation (3.4), we find the 5-dimensional field equation to be invariant. We will study this behavior in more detail only for the 4-dimensional equations. In the moving system the field strengths have the components

$$B_{m'} = \left\{\frac{1}{\mathcal{R}\sin\eta}, 0, 0, -\frac{i}{\mathcal{R}}\right\}, \quad C_{m'} = \left\{\frac{1}{\mathcal{R}\sin\eta}, \frac{1}{\mathcal{R}\sin\eta}\cot\vartheta, 0, -\frac{i}{\mathcal{R}}\right\}, \quad E_{m'} = \left\{\alpha E_1, 0, 0, -i\alpha v E_1\right\}. \tag{3.8}$$

The covariant derivatives of tensors transforms as

$$\Phi_{m'||n'} = L^{m\ n}_{m'n'} \Phi_{m||n} = \left[\Phi_{m'|n'} - L_s^{s'} L^{s}_{m'|n'} \Phi_{s'}\right] - L^{m\ n\ s'}_{m'n's} A_{nm}{}^{s}\Phi_{s'} . \tag{3.9}$$

The Lorentz transformation is a rotation in the tangent space of the surface. The Ricci rotation coefficients (2.6) are made up of the curvatures of the surface. As the curvatures are invariant under rotations in the tangent space, the Ricci rotation coefficients behave like *tensors* under Lorentz transformations. Thus, we write

$$A_{m'n'}{}^{s'} = L_{m'n's}^{m\ n\ s'} A_{mn}{}^{s} \ . \tag{3.10}$$

If we define

$$\Phi_{m'\underset{1}{||}n'} = \Phi_{m'|n'} - L_{n'm'}{}^{s'} \Phi_{s'}, \quad L_{n'm'}{}^{s'} = L_{s}^{s'} L_{m'|n'}^{s} \tag{3.11}$$

as covariant derivative and extend it to the graded derivatives

$$\Phi_{m'\underset{2}{||}n'} = \Phi_{m'|n'} - L_{n'm'}{}^{s'} \Phi_{s'}, \quad \Phi_{m'\underset{3}{||}n'} = \Phi_{m'|n'} - L_{n'm'}{}^{s'} \Phi_{s'} - B_{n'm'}{}^{s'} \Phi_{s'}$$
$$\Phi_{m'\underset{4}{||}n'} = \Phi_{m'|n'} - L_{n'm'}{}^{s'} \Phi_{s'} - B_{n'm'}{}^{s'} \Phi_{s'} - C_{n'm'}{}^{s'} \Phi_{s'} \tag{3.12}$$

and re-examine the field equations, we obtain with

$$m_{n'} = \{\alpha, 0, 0, -i\alpha v\}, \quad u_{n'} = \{i\alpha v, 0, 0, \alpha\} \tag{3.13}$$

Eqs. (2.12) with primed indices. This shows that also the subequations of the Einstein field equations are invariant under Lorentz transformations. This is in accordance with the tensorial properties of the Ricci rotation coefficients as these equations describe the curvatures of the space. The primed equations are still the equations for the static universe envisaged by the moving observers. To obtain information on the forces experienced by the moving observers themselves, we have to introduce a reference system by

$$'m_{m'} = \{1, 0, 0, 0\}, \quad 'u_{m'} = \{0, 0, 0, 1\} \tag{3.14}$$

fixed to these observers. Analyzing the Lorentz term (3.11), we obtain

$$L_{m'n'}{}^{s'} = U_{m'n'}{}^{s'} + G_{m'n'}{}^{s'}$$
$$U_{m'n'}{}^{s'} = 'm_{m'} U_{n'}\, 'm^{s'} - 'm_{m'}\, 'm_{n'} U^{s'}, \quad G_{m'n'}{}^{s'} = u_{m'} G_{n'} u^{s'} - u_{m'} u_{n'} G^{s'} \ . \tag{3.15}$$
$$U_{n'} = \left\{0, 0, 0, -\frac{i}{\mathcal{R}}\right\}, \quad G_{n'} = \{\alpha G_1, 0, 0, -i\alpha v G_1\}$$

G_1 is *numerically equal* to the force E_1 derived in Sec. 2. It results from (2.8) that

$$G_{m'n'}{}^{s'} + E_{m'n'}{}^{s'} = 0 \ . \tag{3.16}$$

This means that the force E deduced from the curvature of the pseudo-hyper sphere is *compensated* by rotations in the tangent space and cannot be measured by observers comoving with the expansion. We emphasize that E and G are two different quantities. E is a 5-dimensional entity, it has a 0-component, which is invariant under Lorentz transformations, while G is a 4-dimensional entity. It has no 0-component and it is generated by rotations in the tangent space. The Lorentz rotations are intrinsically connected with the geometry by $\cos\eta\cos i\chi = 1$. Regarding (3.16) we can re-express the constituents of the Ricci rotation coefficients. Dropping the primes, we obtain

$$A_{mn}{}^{s} = {}^{*}B_{mn}{}^{s} + {}^{*}C_{mn}{}^{s} + D_{mn}{}^{s}, \quad D_{mn}{}^{s} = u_n D_m{}^{s} - u^s D_{mn}, \quad D_{11} = D_{22} = D_{33} = -\frac{i}{\mathcal{R}} \,. \tag{3.17}$$

The last terms are the second fundamental forms of an expanding 3-surface. They are forces acting in the three spacelike directions on any comoving observers independently of the position of these observers. They describe the expansion of the universe. The *B and *C are the spacelike parts of the quantities (2.8). Defining the spacelike covariant derivative as

$$\Phi_{m\wedge n} = \Phi_{m|n} - {}^{*}B_{nm}{}^{s}\Phi_s - {}^{*}C_{nm}{}^{s}\Phi_s \tag{3.18}$$

we obtain

$$R_{mn} = {}^{*}R_{mn} - \left[D_{mn\wedge s}u^s + D_{mn}D_s{}^{s}\right] + 2u_n D_{[s}{}^{s}{}_{\wedge \underline{m}]} - u_m u_n\left[D_s{}^{s}{}_{\wedge r}u^r + D_{sr}D^{sr}\right] \tag{3.19}$$

$\underline{m}$ =1,2,3. As

$$D_{[s}{}^{s}{}_{\wedge \underline{m}]} = 0, \quad D_{mn\wedge s}u^s = 0 \tag{3.20}$$

and also

$${}^{*}B_{n|4} + {}^{*}B_{sn}{}^{r}D_r{}^{s} = 0, \quad {}^{*}C_{n|4} + {}^{*}C_{sn}{}^{r}D_r{}^{s} = 0 \tag{3.21}$$

which can be extracted from *R in (3.19), we are left with

$$\begin{aligned} R_{mn} &= {}^{3}R_{mn} - D_{mn}D_s{}^{s} - u_m u_n D_{sr}D^{sr} \\ {}^{3}R_{mn} &= -\left[{}^{*}B_{n|\underline{m}} + {}^{*}B_n{}^{*}B_m\right] - b_m b_n\left[{}^{*}B^{s}{}_{|s} + {}^{*}B^{s}{}^{*}B_s\right] \\ &\quad -\left[{}^{*}C_{n|\underline{m}} - {}^{*}B_{mn}{}^{s}{}^{*}C_s + {}^{*}C_n{}^{*}C_m\right] - c_m c_n\left[{}^{*}C^{s}{}_{|s} + {}^{*}B_s{}^{*}C^s + {}^{*}C^{s}{}^{*}C_s\right] \end{aligned} \,. \tag{3.22}$$

As all brackets in (3.22) vanish, we obtain from the Einstein equations the relation

$$-\left[D_{mn}D_s{}^{s} + u_m u_n D_{sr}D^{sr} - \frac{1}{2}g_{mn}\left(D_s{}^{s}D_r{}^{r} + D_{sr}D^{sr}\right)\right] + 2A_{m[s}A_{n]}{}^{s} - g_{mn}A_{[r}{}^{s}A_{s]}{}^{r} = 0 \,, \tag{3.23}$$

which is satisfied by (3.17) and (2.14), the two kinds of fundamental forms. Thus, the field equations for the expanding de Sitter model decouple to

$$
\begin{aligned}
&{}^*B_{n|\underline{m}} + {}^*B_n {}^*B_m = 0, \quad {}^*B^s{}_{|s} + {}^*B^s {}^*B_s = 0 \\
&{}^*C_{n|\underline{m}} - {}^*B_{mn}{}^s {}^*C_s + {}^*C_n {}^*C_m = 0, \quad {}^*C^s{}_{|s} + {}^*B_s {}^*C^s + {}^*C^s {}^*C_s = 0 \qquad . \\
&{}^*B_{n|4} + {}^*B_{sn}{}^r D_r{}^s = 0, \quad {}^*C_{n|4} + {}^*C_{sn}{}^r D_r{}^s = 0, \quad D_{[s}{}^s{}_{\wedge \underline{m}]} = 0, \quad D_{mn\wedge s} u^s = 0
\end{aligned}
\tag{3.24}
$$

From Eq. (3.22) we conclude that the 3-dimensional Ricci vanishes. It is easy to proof that the 3-dimensional Riemann vanishes too. At a glance it seems that the de Sitter model contradicts the strategy (II) and (III): Whether the space is curved or is not curved depends on the choice of observers. A more thorough investigation shows that the de Sitter model is also in accordance with (II) and (III). Analyzing one of the equations (2.10), namely

$$B_{a\underset{2}{|||}b} + B_a B_b = 0 \,,$$

which is also valid for the system of expanding observers and using primes again, we get

$$B_{1'|1'} - M_{1'1'}{}^{0'} B_{0'} - L_{1'1'}{}^{4'} B_{4'} + B_{1'} B_{1'} = 0, \quad M_{1'1'}{}^{0'} = -m_{1'} m_{1'} M^0 \ .$$

We see that the sum of the curvature term M and the Lorentz term L vanishes. Thus, the spacelike part of the geometry appears to be flat. Extending this procedure to all other components of the field strengths we finally arrive at Eq. (3.23), which we like to call the equation of compensation. A similar problem was treated for freely falling observers in Schwarzschild geometry [6], where Lorentzian effects compensate the force of gravity. For the accelerated observers the spacelike part of the geometry appears to be flat, but the observers experience tidal forces stretching and squeezing them. These forces can be derived from the second fundamental forms of a shrinking surface.

4. CONCLUSIONS

We have shown that the de Sitter universe, based on a pseudo-hyper sphere, has invariant curvature for any slice and any observer. From an expanding observer the curvature is hidden by Lorentzian effects and cannot be measured. For such an observer the space *appears* to be flat.

5. References

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