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WORMHOLES AND ISOTROPIC CO-ORDINATES

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Contents

We investigate the Einstein-Rosen bridge in Einstein-Rosen co-ordinates and isotropic co-ordinates. We face the traversability of bridges and we discuss the possibility of the second sheet of the bridge to be a black hole.

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1. INTRODUCTION

The static spherically symmetric solution of the Einstein field equations found by Schwarzschild is written usually in the form given by Hilbert. Nowadays it is called the standard form of the Schwarzschild metric. This metric has a singularity at $r = 2M$. To date it has been discussed whether this locus is a physical singularity or a mere co-ordinate singularity. The inner region $0 \leq r < 2M$ of the metric is quoted mostly for the description of black holes. Possibilities had been considered if and how particles can intrude into the inner region. Concerning the discussion on that topic we refer to the papers of Gautreau [1,2,3] and Gautreau and Hoffmann [4], de Sabbata, Pavšič and Recami [5], Janis [6], Cavalleri and Spinelli [7,8], Janis [9], Jaffe and Shapiro [10-13], McGruder III [14], Baierlein [15], Tereno [16], Mitra [17,18], Tereno [19], Mitra [20,21], Shapiro-Teukolsky [22], Crawford and Tereno [23], Krori and Paul [24], Lynden-Bell und Katz [25], Salzmann and Salzmann [26], Logunov, Mestverishvili and Kiselev [27], Loinger [28], and de Sabbata and Shah [29]. A detailed criticism of all attempts to travel beneath the event horizon was given by Mitra [30]. Recently Popławski [31] has published a paper on the Einstein-Rosen bridge by the use of isotropic co-ordinates. Although he concedes that the velocity of infalling particles would be the velocity of light at the event horizon he notes that particles continue moving after reaching the event horizon and that they could pass the bridge. In Sec. 2 we treat the question of traversability of the Einstein-Rosen bridge. In Sec. 3 we present more on isotropic co-ordinates and the Einstein-Rosen bridge.

2. EINSTEIN-ROSEN BRIDGE

Weyl [32] was the first to consider both roots of the Schwarzschild parabola. He noted that the Schwarzschild space is covered twice by the co-ordinate system. He interpreted the first sheet as responsible for the exterior, the second for the interior field. He discovered the isotropic co-ordinates and made the just-mentioned identification once more in isotropic co-ordinates.

Einstein and Rosen [33] introduced the term bridge for the sphere connecting the two sheets at r = 2M. They ascribed the mass to this sphere. Thus, they claimed to have obtained a solution with mass without introducing a stress-energy tensor. They identified the sphere with a neutral particle.

Einstein and Rosen also considered a charged metric with zero mass parameter M. Actually this metric is the Reissner-Nordström metric [34,35]. Thus, the geometry is made up by the charge. In order to obtain a reasonable theory they had to reverse the sign of the electric stress-energy tensor. Later on Som, Santos, and Teixeira [36] showed that M = 0 contradicted the expression for the effective mass for a charged particle.

In a footnote Einstein and Rosen considered to change the sign of the charge term in the metric in order to correct the sign of the stress-energy tensor. Nowadays such a parameter is called l, the NUT parameter and stands for a gravimagnetic monopole. If Einstein and Rosen had worked out this ansatz they would have found the seed metric for the NUT model.

Later on, the two sheets have been interpreted as two remote regions of the universe or as parallel universes connected by the bridge. The latter was called wormhole. In this case the geometry is massless and M is the form parameter of Flamm's paraboloid. No one would claim that anybody could travel through a massive bridge. For M = 0 the Schwarzschild parabola flattens out. The wormhole is closed and the two sheets coincide.

In the literature one finds different views on the curvature of space-time. (i) the term curvature is used for any space with non-Euclidean geometry. A higher dimensional space as embedding space for our 4-dimensional world is not assumed to exist. It is not the space that is curved, but the geometry of the space. This concept of curvature has been impressively formulated by Whittaker [37]. (ii) the curvature is explained by embeddings of surfaces into a higher dimensional flat space. The advantage of this methodology is that all tools of the differential geometry – e. g. the theorems of Gauss and Codazzi – can be used. Evidently, bridges and wormholes are geometrical objects and should be treated with respect to the view (ii) of curvature.

Firstly, we have to note that in discussing bridges and wormholes the 'inner region' $0 \le r < 2M$ of the Schwarzschild metric is definitely excluded. In addition, all values of the curvature quantities in that region are excluded as well.

Many authors use the Kretschmann scalar to judge the geometrical validity for a range of variables. Since we finally have to discuss a bridge we have to take literally all terms related to the curvatures of the space. One has to take notice that the Riemann which should describe the surface seemingly has reasonable values in regions where the surface does not exist if one chooses 'suitable' variables. An arbitrary nonvanishing component of the Riemann tensor, e. g.

$$R^{12}{}_{12} = \frac{M}{r^3} \tag{2.1}$$

seems to be valid for all $0 < r \le \infty$. Inserting the curvature radius of the Schwarzschild parabola $\rho = \sqrt{2r^3/M}$ into (2.1) one obtains

$$R^{12}{}_{12} = \frac{2}{\rho^2} \ . \tag{2.2}$$

Since ρ has its smallest value at the vertex of the parabola, namely $\rho = 4M$, $R^{12}{}_{12}$ does not exist for $r < 2M$. The same is valid for all other components of the Riemann and thus for the Kretschmann scalar

$$K = \frac{48M^2}{r^6} = \frac{192}{\rho^4} \ , \tag{2.3}$$

having a finite value and its highest value at the event horizon as well.

We add some remarks on the Einstein-Rosen paper. We do this only in short, without treating in detail the geometrical properties. One of the aims of Einstein and Rosen has been to remove the singularity at r = 2M from the metric. With a new variable

$$u^2 = r - 2M \tag{2.4}$$

the Schwarzschild metric is free from singularities. It should not be overlooked that u = 0 is equivalent to r = 2M, the definition of the Schwarzschild radius. With the choice of the new co-ordinate system the range of r is limited to $2M \le r \le \infty$. The 'inner region' which is usually quoted for the description of black holes is excluded in the framework of wormholes. In order to clarify the variable u, we rescale it with $u = R/\sqrt{8M}$. Thus, we obtain

$$R^2 = 8M(r - 2M) \; , \tag{2.5}$$

the well-known formula for the Schwarzschild parabola. Rotation of the Schwarzschild parabola creates Flamm's paraboloid. R and r are the Cartesian co-ordinates of the flat embedding space. Einstein and Rosen admit both roots of (2.5)

$$R = \pm\sqrt{8M(r - 2M)} \; . \tag{2.6}$$

The range of R is $-\infty \le R \le \infty$.

The geometry is mirrored, the Einstein-Rosen bridge arises. The metric free from singularities reads as

$$ds^2 = \frac{R^2 + 16M^2}{16M^2} dR^2 + \left(\frac{R^2 + 16M^2}{8M}\right)^2 d\Omega^2 - \frac{R^2}{R^2 + 16M^2} dt^2 \; . \tag{2.7}$$

R = 0 is the vertex of the parabola. At this position the radial line element of both branches of the parabola is dR, and the redshift factor vanishes. Outside Flamm's paraboloid the geometry is not defined. From (2.7) we read the radial tangent vector

$$dx^1 = \sqrt{\frac{R^2 + 16M^2}{16M^2}} dR = \frac{1}{\sin\eta} dR, \quad \eta = \eta(R) \tag{2.8}$$

and we obtain the trigonometric functions

$$\sin\eta = \frac{4M}{\sqrt{R^2 + 16M^2}}, \quad \cos\eta = \frac{R}{\sqrt{R^2 + 16M^2}}, \quad \tan\eta = \frac{4M}{R} \; . \tag{2.9}$$

The value of the velocity of a freely falling observer is identified with $\sin\eta$. According to (2.9) the value of the observer's velocity at the event horizon $R = 0$ is the velocity of light. The curvature radius of the Schwarzschild parabola and the force of gravity are

$$\rho = \frac{\sqrt{\left(R^2 + 16M^2\right)^3}}{16M^2}, \quad E_1 = -\left(\frac{16M^2}{R^2 + 16M^2}\right)^{\frac{3}{2}} \frac{1}{R} = -\frac{1}{R}\sin^3\eta \; . \tag{2.10}$$

An inspection of these formulae shows that quantities of the bridge such as the velocity, the curvature radius, and the force of gravity exhibit the same properties as the quantities of the standard Schwarzschild system. All these properties indicate the hopelessness of an observer travelling through the bridge, if one requires him to keep the laws of the relativity theory. Furthermore, Morris and Thorne [38] showed in a paper on traversable and not traversable wormholes that the Einstein-Rosen model does not belong to the traversable ones. They analyzed the tidal forces near the bridge which are so strong in the environment of the bridge that they would destroy any object.

3. ISOTROPIC CO-ORDINATES

In order to get a deeper insight into the Einstein-Rosen bridge we investigate the features of the isotropic co-ordinate system and we apply them to the bridge. With the help of the regular nonlinear transformation

$$r = \left(1 + \frac{M}{2\bar{r}}\right)^2 \bar{r} \ , \tag{3.1}$$

wherein r is the radial Schwarzschild standard co-ordinate one obtains the line element

$$ds^2 = \left(1 + \frac{M}{2\bar{r}}\right)^4 \left(d\bar{r}^2 + \bar{r}^2 d\vartheta^2 + \bar{r}^2 \sin^2\vartheta d\varphi^2\right) - \left(\frac{1 - \frac{M}{2\bar{r}}}{1 + \frac{M}{2\bar{r}}}\right)^2 dt^2 \tag{3.2}$$

in isotropic co-ordinates which is valid for all $\bar{r}$ of $0 < \bar{r} \le \infty$. With respect to the application to wormholes one has to analyze whether the isotropic representation basically differs from the standard representation of the Schwarzschild model. Taking a glance at the metric one recognizes that the redshift factor vanishes at

$$\bar{r}_H = \frac{M}{2} \ . \tag{3.3}$$

The radial standard Schwarzschild differential and the isotropic one are related by

$$dr = \left(1 - \frac{M^2}{4\bar{r}^2}\right) d\bar{r} \ . \tag{3.4}$$

With $dr/d\bar{r} = 0$ it can be shown that the function $r(\bar{r})$ has a minimum at the event horizon. Within the range $0 \le \bar{r} < \bar{r}_H$ the variable r preserves values $> 2M$. If $\bar{r}$ decreases in this region the standard variable r increases. In particular $\bar{r} = 0$ corresponds to $r = \infty$. Within the co-ordinate range $\bar{r}_H < \bar{r} \le \infty$ both variables r and $\bar{r}$ simultaneously increase. If the metric (3.2) is still another notation for the Schwarzschild metric the Schwarzschild ‘inner region’ $0 \le r < 2M$ is excluded by the isotropic co-ordinates from the outset. No value of $\bar{r}$ corresponds to a value of r beneath the event horizon. Therefore the isotropic co-ordinates rule out black holes as well. Two sheets of the space corresponding to the two ranges of the isotropic co-ordinates

$$0 \le \bar{r} < \bar{r}_H, \quad \bar{r}_H < \bar{r} \le \infty \ . \tag{3.5}$$

are described by (3.2).

The standard co-ordinate differential dr described by (3.4) is positive on the upper sheet. The tangent vector of the parabola is pointing outwards. In contrast, dr is negative in the lower sheet. The tangent vector is pointing inwards. The orientation of the tangent vector does not make a jump at the vertex of the parabola. Both sheets are parts of a wormhole (Einstein-Rosen bridge) separated by the bridge. Alternatively the two sheets can be identified.

The basic quantities of the model expressed in isotropic co-ordinates are calculated straight forward but one has carefully to interpret the results. It is mentioned in the literature that the isotropic co-ordinates do not faithfully represent distances. This deficiency can be removed by locally gauging the rods used for measuring distances. Resolving (3.1) for $\bar{r}$ one obtains the two roots

$$\bar{r}_+ = \frac{1}{2}\left(r - M + \sqrt{r^2 - 2Mr}\right), \quad \bar{r}_- = \frac{1}{2}\left(r - M - \sqrt{r^2 - 2Mr}\right) \tag{3.6}$$

and the rules for gauging the rods for measurements on the sheets. Furthermore, the asymmetry of the upper and lower sheets that could appear plainly using ungauged isotropic co-ordinates is removed [1]. The use of isotropic co-ordinates has the advantage that the Schwarzschild inner region is definitely excluded. Moreover, one does not obtain values for curvature quantities where these do not exist. In isotropic co-ordinates the Kretschmann scalar is regular for all $\bar{r}$ and reads as

$$K = R^{mnsr}R_{mnsr} = \frac{48M^2}{\bar{r}^6}\frac{1}{\left(1+\frac{M}{2\bar{r}}\right)^{12}} \ . \tag{3.7}$$

Correctly gauging the co-ordinate scales the Kretschmann has the same values for both sheets and the same values as the Kretschmann in the standard co-ordinates. Moreover, the Kretschmann does not show evidence of values where it does not exist. In addition, it has no values corresponding to the Schwarzschild inner region at the outset.

In isotropic co-ordinates the value of the velocity of a freely falling observer is

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[1] The Schwarzschild parabola $R^2 = 8M(r - 2M)$ is form invariant under the transformation

$$M = 4\bar{M}, \quad r = \left(1 + \frac{2\bar{M}}{\bar{r}}\right)^2 \bar{r}, \quad R = \frac{2\bar{R}^2}{\sqrt{\bar{R}^2 + 16\bar{M}^2}} \ ,$$

The $\{R, r\}$ are the Cartesian co-ordinates of the flat embedding space. The $\{\bar{R}, \bar{r}\}$ can be interpreted as rectangular co-ordinates as well if a space-dependent gauging of the rods measuring the geometrical quantities is admitted. The event horizon is located at $\bar{r}_H = 2\bar{M}$.

$$v(\bar{r}) = \frac{1}{1+\frac{M}{2\bar{r}}}\sqrt{\frac{2M}{\bar{r}}} \ . \tag{3.8}$$

At the event horizon the velocity grows to be $v(\bar{r}) = 1$, the velocity of light in natural units. This corresponds to the maximum of the value of the function $v(\bar{r})$. The function of the isotropic velocity decreases after $\bar{r}$ has run through $\bar{r}_H$.

Reading the tetrads from (3.2) and evaluating the Ricci-rotation coefficients one obtains the field quantities

$$A_{21}{}^{2} = A_{31}{}^{3} = \frac{1-\frac{M}{2\bar{r}}}{\left(1+\frac{M}{2\bar{r}}\right)^{3}}\frac{1}{\bar{r}}, \ A_{32}{}^{3} = \frac{1}{\left(1+\frac{M}{2\bar{r}}\right)^{2}}\frac{1}{\bar{r}}\cot\vartheta, \ A_{41}{}^{4} = \frac{1}{\left(1+\frac{M}{2\bar{r}}\right)^{3}\left(1-\frac{M}{2\bar{r}}\right)}\frac{M}{\bar{r}^{2}} = -E_1 \ . \tag{3.9}$$

satisfying the vacuum field equations.

The last relation shows that the force of gravity E becomes infinitely large at the Schwarzschild radius, as expected. The standard representation and the isotropic representation differ substantially. The Schwarzschild force of gravity becomes imaginary in the inner region, the isotropic force is real and attractive in both sheets[2].

There is more insight into the behavior of the force of gravity if we use the angle of ascent $\eta = \eta(\bar{r})$ of the Schwarzschild parabola. Remembering the geometrical structure of the Schwarzschild model we write $\sin\eta(\bar{r}) = v(\bar{r})$. If we evaluate the curvature radius of the Schwarzschild parabola in isotropic co-ordinates

$$\rho = \left(1+\frac{M}{2\bar{r}}\right)^{3}\sqrt{\frac{2\bar{r}^{3}}{M}} \ . \tag{3.10}$$

we lastly obtain

$$E_1 = -\frac{1}{\rho}\tan\eta \ , \tag{3.11}$$

the same expression that is valid for the standard representation of the Schwarzschild model [39]. In addition, one immediately obtains with the help of the transformation (3.1), the connexion coefficients for the Schwarzschild standard form in tetrad representation

$$\frac{1}{r}\cos\eta, \quad \frac{1}{r}\cos\eta, \quad \frac{1}{r}\cot\vartheta, \quad -\frac{1}{\cos\eta}\frac{M}{r^{2}}, \quad \cos\eta(r) = \sqrt{1-2M/r} \ . \tag{3.12}$$

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[2] Obviously, the quantities (3.9) have different signs in the two sheets. The reason is that the scale of $\bar{r}$ is retrograde in the lower sheet and the signs of dr are different. On the two sheets the tangent vectors have opposite orientations. Identifying the two sheets one has to reverse the signs and the local directions of the lower sheet.

Having calculated all the basic quantities and basic relations of the model in isotropic co-ordinates, we find in accordance with the standard representation of the model and the Einstein-Rosen bridge that all these quantities have the same properties at the Schwarzschild radius. At this locus, any massive object falling towards the center of gravity, reaches the velocity of light, and the gravitational force is infinitely large. This is valid for both sheets of the bridge. The bridge is not traversable.

Popławski identifies the region $M/2 < \bar{r} \leq \infty$ with the exterior field of a Schwarzschild black hole and $0 \leq \bar{r} < M/2$ as the interior of the black hole. The latter is regarded by him as the image of the exterior sheet. He performs a further co-ordinate transformation

$$r' = \frac{M^2}{4\bar{r}} \tag{3.13}$$

leaving the form of the metric (3.2) invariant. This new co-ordinate has to be gauged in a similar way as $\bar{r}$. He argues that 'an infalling radial geodesic motion inside a black hole appears in terms of the new radial co-ordinate r' as an outgoing motion from a white hole'.

Evidently, the region $0 \leq \bar{r} < M/2$ could suggest that the region for an assumed black hole is a finite region. On the other hand this region corresponds to the unlimited region $2M < r \leq \infty$ on the second sheet of the Einstein-Rosen bridge. This is a contradiction because no co-ordinate transformation can alter the geometric structure of a model.

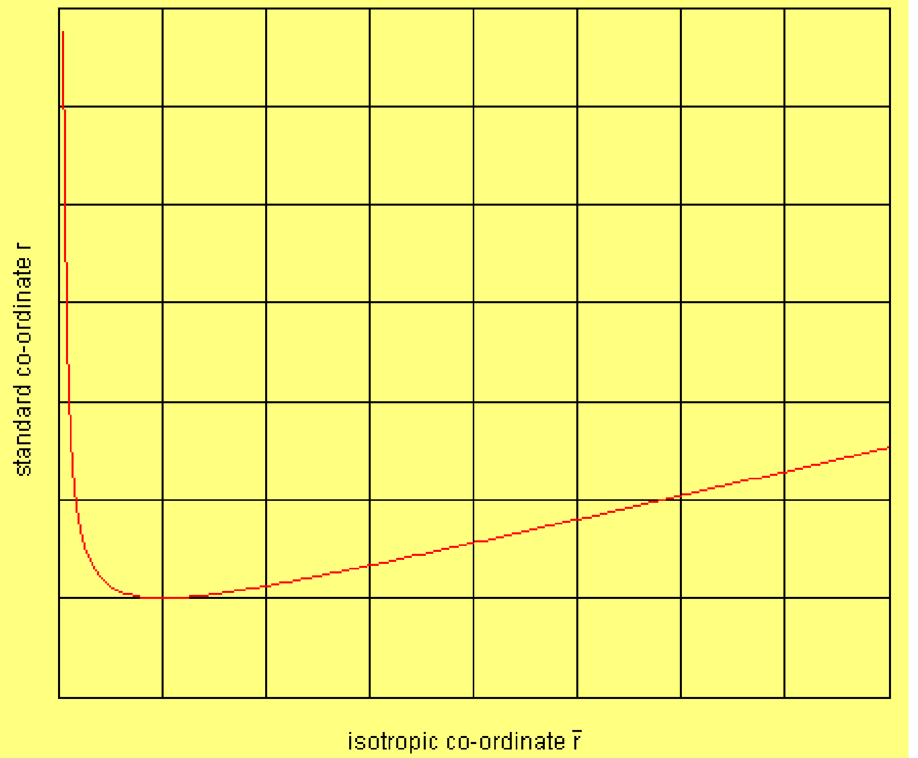

Fig 1.

The reason of this ambiguity is that isotropic co-ordinates are not faithful (Fig. 1). They are retrograde on the second sheet. Not gauging the co-ordinate axes nonlinearly with different gauges on the two sheets isotropic co-ordinates exhibit asymmetric features with respect to these two sheets. The use of isotropic co-ordinates might lead to misconceptions.

4. CONCLUSIONS

It is generally accepted that the equations of a gravitation model must be invariant under co-ordinate transformations. We examined the Schwarzschild model in three different co-ordinate systems, namely in Schwarzschild-standard co-ordinates, in isotropic co-ordinates, and in Einstein-Rosen co-ordinates. We emphasized particularly the behavior of the force of gravity and the velocity of a freely falling observer at the event horizon. At this position, the value of the velocity of a freely falling observer takes the value of the velocity of light in all three systems. The force of gravity blows up as well. It is to be concluded that the event horizon cannot be exceeded, either into a black hole or via the bridge of a wormhole. Further, it must be mentioned that the Schwarzschild inner region cannot be described either with the isotropic co-ordinate system or with the Einstein-Rosen co-ordinate system. If one considers that statements can be made on the equations of a gravitational model independently of the co-ordinate choice, the Schwarzschild inner region must be excluded as a possible piece of the Schwarzschild theory. Therefore there is no evidence for black holes.

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